COMBINATORICS OF BIFURCATIONS IN EXPONENTIAL PARAMETER SPACE

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ABSTRACT. We give a complete combinatorial description of the bifurcation structure in the space of exponential maps $z \mapsto \exp(z) + \kappa$. This combinatorial structure is the basis for a number of important results about exponential parameter space. These include the fact that every hyperbolic component has connected boundary [RS, S3], a classification of escaping parameters [FRS], and the fact that all dynamic and parameter rays at periodic addresses land [R2, S1].

CONTENTS

1. INTRODUCTION

Ever since Douady and Hubbard's celebrated study of the Mandelbrot set [DH], combinatorics has played a fundamental role for the dynamics of complex polynomials. In particular, the concept of *external rays*, both in the dynamical and parameter plane, and the landing behavior of such rays, has helped in the understanding of polynomial Julia sets and bifurcation loci. This program has been particularly successful for the simplest

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polynomial parameter spaces: the quadratic family $z \mapsto z^2 + c$ and its higher-dimensional cousins, the unicritical families $z \mapsto z^d + c$ [DH, S4, ES].

In this article, we consider the space of exponential maps,

$$E_\kappa : \mathbb{C} \to \mathbb{C}; z \mapsto \exp(z) + \kappa.$$

It is well-known that a restriction on the number of *singular values* (i.e., critical and asymptotic values) of an entire function generally limits the amount of different dynamical features that can appear for the same map. Since exponential maps are the only transcendental entire functions which have just one singular value, namely the omitted value κ (see e.g. [M2, Appendix D]), exponential maps form the simplest parameter space of entire transcendental functions. In addition, the exponential family can be considered as the limit of the polynomial unicritical families, and thus is an excellent candidate to apply the combinatorial methods which were so successful for Mandel- and Multibrot sets.

Recently, some progress has been made in this direction: a complete classification of escaping points for exponential maps in terms of *dynamic rays* was given in [SZ1], and a similar construction was carried out to obtain *parameter rays* [F, FS]. Also, exponential maps with attracting periodic cycles were classified in [S2] using combinatorics.

Nonetheless, a basic description of exponential dynamics in analogy to the initial study of the Mandelbrot set should involve at least the following results.

(a) For every hyperbolic component W, there is a homeomorphism of pairs $(W, \overline{W}) \to (\mathbb{H}, \overline{\mathbb{H}})$, where $\mathbb{H}$ is the left half plane. (In particular, ∂W is a Jordan curve.)
(b) Every periodic parameter ray lands at a parabolic parameter.
(c) If the singular value of E_κ does not escape to ∞, then all periodic dynamic rays of E_κ land.
(d) If the singular value of E_κ does not escape to ∞, then every repelling periodic point of E_κ is the landing point of a periodic dynamic ray.

For unicritical polynomials, the analogs of these statements all have relatively short analytic proofs (see e.g. [PR] for (b) and [M1, Theorems 18.10 and 18.11] for (c) and (d)), but these break down in the exponential case. Nonetheless, it is possible to prove items (a) through (c), using a novel approach based on a thorough study of parameter space. One of the goals of this article is to provide the first ingredients in this approach by obtaining a complete description of the combinatorial structure of parameter space (as given by bifurcations of hyperbolic components). In the sequel [RS], this description is used to prove (a), which, in turn, leads to proofs of (b) [S1] and (c) [R2], as well as some progress on (d) [R2].

To illustrate the difficulties we face, let us consider the structure of child components bifurcating from a given hyperbolic component. If we already knew results (a) and (b) above, it would be quite easy to obtain the following description; compare Figure 1.

Let W be a hyperbolic component of period $n \geq 2$, and let $\mu : W \to \mathbb{D}^*$ be the multiplier map (which maps each hyperbolic parameter to the multiplier of its unique attracting cycle). Then there exists a conformal isomorphism $\Psi_W : \mathbb{H} \to W$ with $\mu \circ \Psi_W = \exp$ which extends continuously to $\partial\mathbb{H}$ and such that $\Psi_W(0)$ (the *root* of W) is the landing point of

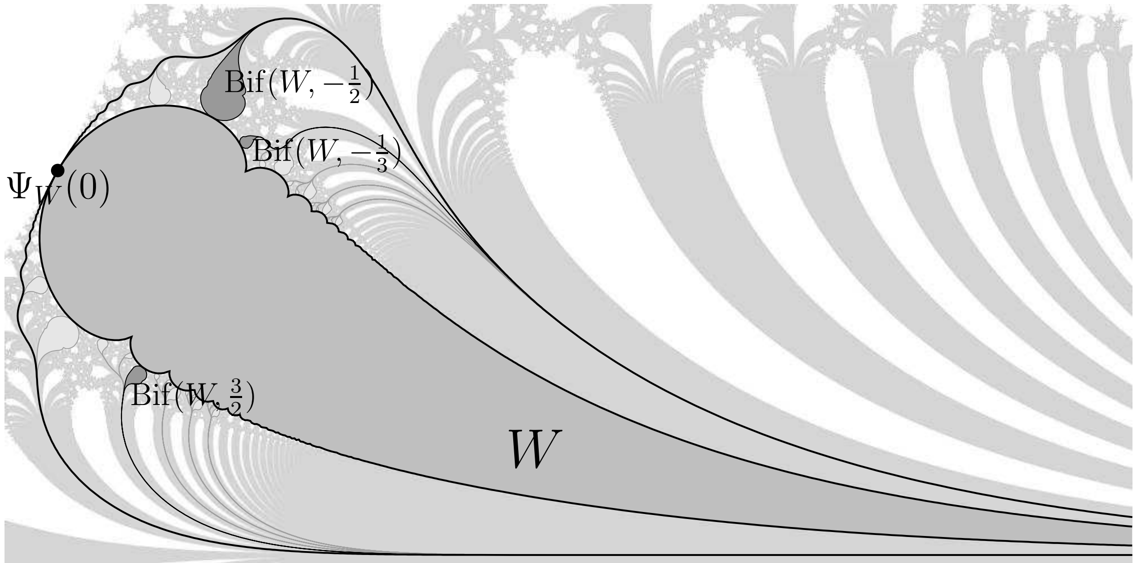

FIGURE 1. Structure of child components bifurcating from a period 3 hyperbolic component in exponential parameter space

two periodic parameter rays. The region containing W which is enclosed by these two rays is called the *wake* of W.

For every $h = \frac{p}{q} \in \mathbb{Q} \setminus \mathbb{Z}$, the point $\Psi_W(2\pi i h)$ is the root point of a (unique) hyperbolic component $\mathrm{Bif}(W, h)$ of period qn (called a *child component* of W). The component $\mathrm{Bif}(W, h)$ tends to infinity above or below W depending on whether $h < 0$ or $h > 0$ (respectively). If $0 < h_1 < h_2$ or $h_1 < h_2 < 0$, then $\mathrm{Bif}(W, h_1)$ tends to infinity below $\mathrm{Bif}(W, h_2)$. Any hyperbolic component other than W which lies in the wake of W is contained in the wake of a unique child component $\mathrm{Bif}(W, h)$.

The problem we face is that, without knowing (a), we do not know that all "expected" bifurcations really exist. Thus we will need to obtain a purely *combinatorial* version of this description (given by Theorem 6.8) without being able to use the topological structure of parameter space. This makes many arguments (and statements) much more delicate.

Another goal of our article is to explain the relation among certain combinatorial objects which appear in exponential dynamics. In particular, there are several such objects associated to any hyperbolic component.

- The *characteristic external addresses* of W (Definition 3.4). These are the addresses of the parameter rays bounding the wake of W.
- The *intermediate external address* of W (Section 2). This is an object which does not appear in Multibrot sets. It describes the combinatorial position of the singular value within the dynamical plane of a parameter in W. At the same time, it describes the position of the hyperbolic component W itself in the vertical structure of parameter space.

- The *kneading sequence* of W (Definition 3.5). This object describes the itinerary of the singular orbit with respect to a natural *dynamical* partition (as opposed to the *static* partition used to define external addresses).
- The *internal address* of W (Definition 7.8). Introduced for Multibrot sets in [LS], this address describes the position of W within the bifurcation structure of hyperbolic components. Its relative, the *angled internal address* (Definition A.5), is decorated with some additional information.

Our study yields algorithms to convert between these different objects (where possible), and also to compute the address of any child component. These algorithms are collected in Appendix B.

Finally, the combinatorial objects and methods used in this article have applications far beyond the scope of our investigation, and are likely to play a significant role in further studies of the exponential family (as they did for the Mandelbrot set). Thus, we aim to present a comprehensive exposition of these concepts which may serve as a reference in the future.

We should emphasize that all results of this article — with the exception of the analytical considerations of Section 5 — are completely combinatorial and could be formulated and proved without any reference to the underlying exponential maps. However, we prefer to carry out an argument within an actual dynamical plane whenever possible, as we find this much more intuitive. (Compare for example the definition of orbit portraits in Section 3, as well as the proofs of Lemma 3.10 and Proposition 7.4).

Since the combinatorial structure of exponential parameter space is a limit of that for unicritical polynomials, it would be possible to infer many of our results from corresponding facts for these families. However, many of these — particularly for Multibrot sets of higher degrees — are themselves still unpublished. Also, there are aspects of exponential dynamics, such as the *intermediate external address* of a hyperbolic exponential map, which would not feature in such an approach. We have thus decided to give a clean self-contained account in the exponential case.

Structure of this article. In the following two sections, we give a comprehensive overview of several combinatorial concepts for exponential maps: external addresses, dynamic rays, intermediate external addresses, orbit portraits, characteristic rays and itineraries. In Section 4, we consider some basic facts about hyperbolic components of exponential maps, and how they are partitioned into *sectors*.

Section 5 is the only part of the article in which analytical considerations are made: we investigate the stability of orbits at a parabolic point, allowing us to understand the structure of bifurcations occurring at such points. This provides the link between our subsequent combinatorial considerations and the exponential parameter plane. While the arguments in this section are very similar to those in the polynomial setting, there are some surprises: the combinatorics of a parent component can be determined with great ease from that of a child component, thanks to the new feature of intermediate external addresses.

With these preliminaries, we will be in a position to prove our main results. Section 6 deals with the structure of (combinatorial) child components of a given hyperbolic component, as discussed above. Section 7 introduces introduces internal addresses, giving a "human-readable" combinatorial structure to parameter space, and shows how they are related to the combinatorical concepts defined before.

In Appendix A, we consider some further concepts. These are not required for the proofs in [RS] but follow naturally from our discussion and will be collected for future reference. Appendix B explicitly collects the combinatorial algorithms which are implied by our results.

For the reader's convenience, a list of notation and an index of the relevant combinatorial concepts is provided at the end of the article.

Some remarks on notation. We have chosen to parametrize our exponential maps as $z \mapsto E_\kappa(z) = \exp(z) + \kappa$. Traditionally, they have often been parametrized as $\lambda \exp$, which is conjugate to E_κ if $\lambda = \exp(\kappa)$. We prefer our parametrization mainly because the behavior of exponential maps at ∞, and in particular the asymptotics of external rays, do not depend on the parameter in this parametrization. Note that this is also the case in the usual parametrization of quadratic polynomials as $z \mapsto z^2 + c$. Also, under our parametrization the picture in the parameter plane reflects the situation in the dynamical plane, which is a conceptual advantage. Note that E_κ and $E_{\kappa'}$ are conformally conjugate if and only if $\kappa - \kappa' \in 2\pi i\mathbb{Z}$. This will prove useful in the combinatorial description. When citing known results, we always translate them into our parametrization.

If $\gamma : [0, \infty) \to \mathbb{C}$ is a curve, we shall say that $\lim_{t\to\infty} \gamma(t) = +\infty$ (or, in short, call γ a *curve to* $+\infty$) if $\operatorname{Re}\gamma(t) \to +\infty$ and $\operatorname{Im}\gamma$ is bounded; analogously for $-\infty$. The n-th iterate of any function f will be denoted by f^n. Whenever we write a rational number as a fraction $\frac{p}{q}$, we will assume p and q to be coprime.

We conclude any proof and any result which immediately follows from previously proved theorems by the symbol $\blacksquare$. A result which is cited without proof is concluded by $\square$.

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2. Combinatorics of Exponential Maps

An important combinatorial tool in the study of polynomials is the structure provided by *dynamic rays*, which foliate the basin of infinity. Similarly, throughout this article, we will assign combinatorics to curves in the dynamical plane of an exponential map, both in the set of escaping points and in Fatou components. This section will review these methods, which were introduced in [SZ1] and [S2].

External addresses and dynamic rays. A sequence $\underline{s} = s_1 s_2 \dots$ of integers is called an (infinite) external address[1]. If $s_1, \dots, s_n \in \mathbb{Z}$, then the address obtained by periodically repeating this sequence will be denoted by $\overline{s_1 \dots s_n}$.

Let $\kappa \in \mathbb{C}$ and let $\gamma : [0, \infty) \to \mathbb{C}$ be a curve in the dynamical plane of E_κ. Then we say that γ has *external address* $\underline{s}$ if and only if

$$\lim_{t\to\infty} \operatorname{Re} E_\kappa^{j-1}(\gamma(t)) = +\infty \qquad \text{and} \qquad \lim_{t\to\infty} \operatorname{Im} E_\kappa^{j-1}(\gamma(t)) = 2\pi s_j$$

for all $j \geq 1$; in this case, we also write $\underline{s} = \operatorname{addr}(\gamma)$. An external address $\underline{s}$ is called *exponentially bounded* if there exists some $x > 0$ such that $2\pi|s_k| < F^{k-1}(x)$ for all $k \geq 1$, where $F(t) = \exp(t) - 1$ is used as a model function for exponential growth.

The *set of escaping points* of E_κ is defined to be

$$I := I(E_\kappa) := \{z \in \mathbb{C} : |E_\kappa^n(z)| \to \infty\}.$$

It is known that the Julia set $J(E_\kappa)$ is the closure of $I(E_\kappa)$ [EL1, EL2]. In [SZ1], the set $I(E_\kappa)$ has been completely classified. In particular, it was shown that it consists of curves to ∞, so-called *dynamic rays*. We will use this result in the following form. (Note that the fact that dynamic rays are the path-connected components of $I(E_\kappa)$ was stated but not proved in [SZ1]; for a proof compare [FRS].)

2.1. Theorem and Definition (Dynamic Rays).
Let $\kappa \in \mathbb{C}$. Then, for every exponentially bounded address $\underline{s}$, there exists a unique injective curve $g_{\underline{s}} : [0, \infty) \to I(E_\kappa)$ or $g_{\underline{s}} : (0, \infty) \to I(E_\kappa)$ which has external address $\underline{s}$ and whose trace is a path-connected component of $I(E_\kappa)$. The curve $g_{\underline{s}}$ is called the dynamic ray *at address $\underline{s}$.*

If $\kappa \notin I(E_\kappa)$, then every path-connected component of $I(E_\kappa)$ is a dynamic ray. If $\kappa \in I(E_\kappa)$, then every such component is either a dynamic ray or is mapped into a dynamic ray under finitely many iterations.

Remark. In order to state this theorem as given, dynamic rays need to be parametrized differently from [SZ1]. In this article, we will only be using dynamic rays at periodic addresses $\underline{s}$, and for these our parametrization agrees with that of [SZ1], provided that the singular orbit does not escape.

Intermediate external addresses. We shall also need to assign combinatorics to certain curves in Fatou components which, under finitely many iterations, map to a curve to $-\infty$. Let $\gamma : [0, \infty) \to \mathbb{C}$ be a curve in the dynamical plane of κ such that, for some $n \geq 1$, $\lim_{t\to\infty} E_\kappa^{n-1}(\gamma(t)) = -\infty$. Then there exist $s_1, \dots, s_{n-2} \in \mathbb{Z}$ and $s_{n-1} \in \mathbb{Z} + \frac{1}{2}$ such that

$$\lim_{t\to\infty} \operatorname{Re}(E_\kappa^{j-1}(\gamma(t))) = +\infty \qquad \text{and} \qquad \lim_{t\to\infty} \operatorname{Im}(E_\kappa^{j-1}(\gamma(t))) = 2\pi s_j$$

for $j = 1, \dots, n-1$. We call

$$\operatorname{addr}(\gamma) := s_1 s_2 \dots s_{n-1}\infty \tag{1}$$

[1] For brevity, we will frequently omit the adjective "external"; "address" will always mean "external address" unless explicitly stated otherwise

the *intermediate external address* of γ. Any sequence of the form (1) with $s_1, \dots, s_{n-2} \in \mathbb{Z}$ and $s_{n-1} \in \mathbb{Z} + \frac{1}{2}$ is called an intermediate external address (of length n).

To illustrate the relationship between infinite and intermediate external addresses, consider the following construction. Define

$$f : \mathbb{R} \setminus \{(2k-1)\pi : k \in \mathbb{Z}\} \to \mathbb{R}, t \mapsto \tan(t/2).$$

Then to any (infinite) external address $\underline{s}$ we can associate a unique point x for which $f^{k-1}(x) \in \big((2s_k - 1)\pi, (2s_k + 1)\pi\big)$ for all k. However, there are countably many points which are not realized by any external address in this way, namely the preimages of ∞ under the iterates of f. Adding intermediate external addresses corresponds to filling in these points. The space $\overline{\mathcal{S}}$ of all infinite and intermediate external addresses is thus order-isomorphic to the circle $\overline{\mathbb{R}} \cong \mathbb{S}^1$. We also set $\mathcal{S} := \overline{\mathcal{S}} \setminus \{\infty\}$. The *shift map* is the function

$$\sigma : \mathcal{S} \to \overline{\mathcal{S}}; s_1 s_2 \dots \mapsto s_2 \dots;$$

note that σ corresponds to the function f in the above model.

Lexicographic and vertical order. The space $\mathcal{S}$ naturally comes equipped with the *lexicographic order* on external addresses. (As seen above, this ordered space is isomorphic to the real line $\mathbb{R}$, and in particular is complete.) Similarly, the space $\overline{\mathcal{S}}$ carries a (complete) circular ordering. In our combinatorial considerations, we will routinely use the following fact.

2.2. Observation (Shift Preserves Order On Small Intervals)**.**
For every $\underline{s} = s_1 s_2 s_3 \dots$ and $\underline{s}' := (s_1 + 1) s_2 s_3 \dots$, the map $\sigma\colon [\underline{s}, \underline{s}') \to \overline{\mathcal{S}}$ preserves the circular order of $\overline{\mathcal{S}}$.

Any family of pairwise disjoint curves to $+\infty$ has a natural *vertical order*: among any two such curves, one is *above* the other. More precisely, suppose that $\gamma : [0, \infty) \to \mathbb{C}$ is a curve to $+\infty$ and define $\mathcal{H}_R := \{z \in \mathbb{C} : \operatorname{Re} z > R\}$ for $R > 0$. If R is large enough, then the set $\mathcal{H}_R \setminus \gamma$ has exactly two unbounded components, one above and one below γ. Any curve $\widetilde{\gamma}$ to $+\infty$ which is disjoint from γ must (eventually) tend to ∞ within one of these.

It is an immediate consequence of the definitions that, if γ and $\widetilde{\gamma}$ have (infinite or intermediate) external addresses $\operatorname{addr}(\gamma) \neq \operatorname{addr}(\widetilde{\gamma})$, then γ is above $\widetilde{\gamma}$ if and only if $\operatorname{addr}(\gamma) > \operatorname{addr}(\widetilde{\gamma})$.

Intermediate address of attracting and parabolic dynamics. Let us suppose that E_κ has an attracting or parabolic periodic point. Then the singular value κ is contained in some periodic Fatou component; we call this component the *characteristic Fatou component*. Let $U_0 \mapsto U_1 \mapsto \dots \mapsto U_n = U_0$ be the cycle of periodic Fatou components, labeled such that U_1 is the characteristic component. (This will be our convention for the remainder of the paper.) Since U_1 contains a neighborhood of the singular value, U_0 contains a left half plane. In particular, U_0 contains a horizontal curve along which $\operatorname{Re}(z) \to -\infty$. Its pullback to U_1 under E_κ^{n-1} has an intermediate external address $\underline{s}$ of length n. (The

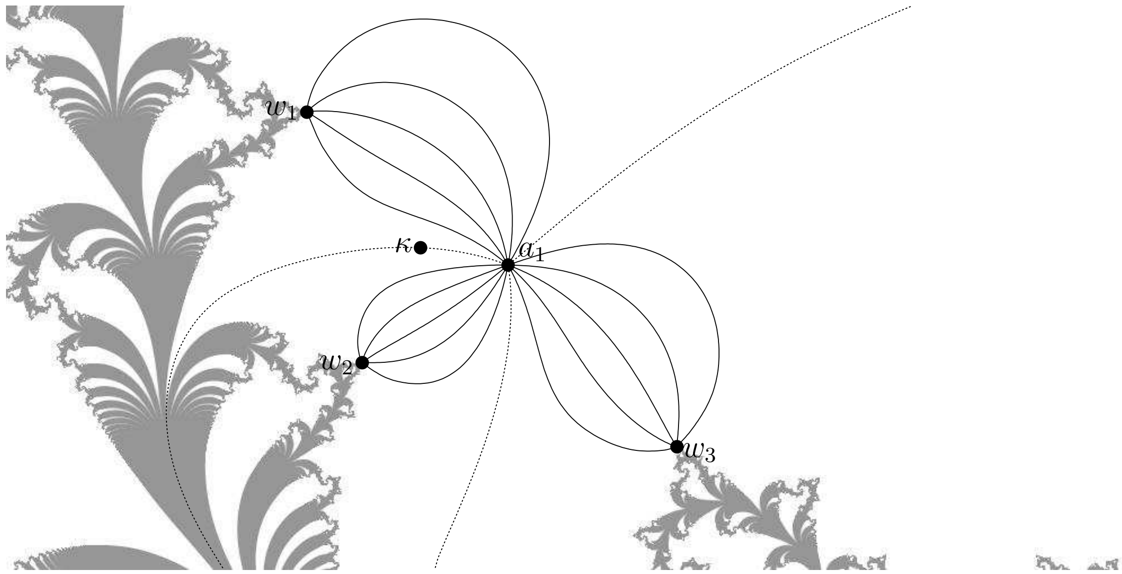


FIGURE 2. Attracting dynamic rays for a parameter where the attracting multiplier has angle 1/3. All unbroken attracting rays land at the distinguished boundary orbit $\{w_i\}$; while the three broken attracting rays (shown as dotted lines) contain the singular orbit.

address $\underline{s}$ does not depend on the initial choice of the curve to $-\infty$, since the latter is unique up to homotopy in U_0.)

We call $\underline{s}$ the *intermediate external address of* κ and denote it by $\operatorname{addr}(\kappa)$; it will play a special role throughout this article. The following was proved independently in [S2] and [DFJ]; the idea of the proof goes back to [BR, Section 7].

2.3. Proposition (Existence of Attracting Maps with Prescribed Combinatorics)**.** *Let $\underline{s}$ be an intermediate external address. Then there exists an attracting parameter κ with* $\operatorname{addr}(\kappa) = \underline{s}$. □

A converse result was also proved in [S2]: the external address $\operatorname{addr}(\kappa)$ determines E_κ up to quasiconformal conjugacy (see Proposition 4.2).

Attracting dynamic rays. We shall frequently have need for a canonical choice of certain curves in a Fatou component. Let E_κ have an attracting orbit, which we label $a_0 \mapsto a_1 \mapsto \dots \mapsto a_n = a_0$ such that $a_i \in U_i$. Note that we can connect κ to a_1 by a straight line in linearizing coordinates. The pullback of this curve under E_κ^n along the orbit of a_1 is then a curve $\gamma \subset U_1$ which connects a_1 to ∞ and has $\operatorname{addr}(\gamma) = \operatorname{addr}(\kappa)$. We call γ the *principal attracting ray* of E_κ. More generally, any maximal curve in U_1 which starts at a_1 and is mapped into a radial line by the extended Kœnigs map $\varphi : U_1 \to \mathbb{C}$ is called an *attracting dynamic ray* of E_κ. Apart from the principal attracting ray, the attracting dynamic ray which contains κ will be particularly important. Those attracting dynamic rays which map to a curve to $-\infty$ in U_0 under an iterate E_κ^{nj-1}, $j > 0$, will sometimes be

called *broken*; the principal attracting ray is one of these. Note that such a broken ray is mapped to a proper subpiece of the attracting ray containing κ by E_κ^{nj}.

It can be shown that every unbroken attracting dynamic ray lands at a point in $\mathbb{C}$ [R1, Theorem 4.2.7]. We shall only need this fact in the case of rational multipliers, where it is simpler to prove (see Figure 2).

2.4. Definition and Lemma (Distinguished Boundary Orbit)**.**
Consider an attracting exponential map E_κ of period n whose attracting multiplier has rational angle $\frac{p}{q}$. Then the orbit of the principal attracting ray of E_κ under E_κ^n consists of q attracting dynamic rays and contains all points of the singular orbit in U_1.

Every other attracting dynamic ray starting at a_1 is periodic of period q and lands at one of the points of a unique period q orbit of E_κ^n on ∂U_1. This orbit is called the distinguished boundary orbit *on ∂U_1.*

Proof. Analogous to the case $q = 1$ [S2, Lemma 6.1]. □

3. Orbit Portraits and Itineraries

Orbit Portraits. Following Milnor [M3], we will use the notion of *orbit portraits* to encode the dynamics of periodic rays landing at common points. As in the case of quadratic polynomials, this is important for understanding the structure of parameter space.

3.1. Definition (Orbit Portrait)**.**
Let $\kappa \in \mathbb{C}$ and let $(z_1, \dots, z_n)$ be a repelling or parabolic periodic orbit for E_κ. Define

$$A_k := \{\underline{r} \in \mathcal{S} : \underline{r} \text{ is periodic and the dynamic ray } g_{\underline{r}} \text{ lands at } z_k\}.$$

Then $\mathcal{O} := \{A_1, \dots, A_n\}$ is called the orbit portrait *of (z_k). The orbit (and the orbit portrait) is called* essential *if $|A_k| > 1$ for any k. An essential orbit portrait is called of* satellite type *if it contains only one cycle of rays; otherwise it is called* primitive*.*

3.2. Lemma (Basic Properties of Orbit Portraits)**.**
Let $\mathcal{O} = \{A_1, \dots, A_n\}$ be an orbit portrait. Then all A_k are finite, and the shift map carries A_k bijectively onto A_{k+1}. Furthermore, all addresses in the portrait share the same period qn (called the ray period *of the orbit) for some integer $q \geq 1$.*

Proof. The proof that all rays share the same period and are transformed bijectively by the shift is completely analogous to the polynomial setting [M1, Lemma 18.12]. Let qn be the common period of the rays in $\mathcal{O}$ and let $\underline{s} = \overline{s_1 s_2 \dots s_{qn}} \in A_1$. It is easy to see that

$$A_1 \subset \left\{\overline{s'_1 s'_2 \dots s'_{qn}} : |s'_k - s_k| \leq 1\right\}$$

(see e.g. [Rr, Lemma 5.2]). The set on the right hand side is finite, as required. ■

An orbit portrait can also be defined as an abstract combinatorial object, without reference to any parameter. We will not do this here, but we will often suppress the actual choice of parameter present in its definition.

Characteristic rays. For quadratic polynomials, every orbit portrait has two distinguished rays, which are exactly the two rays which separate the critical value from all other rays in the portrait [M3, Lemma 2.6]. A corresponding statement for exponential maps is given by the following result.

3.3. Definition and Lemma (Characteristic Rays [S2, Lemma 5.2])**.**
Let $\kappa \in \mathbb{C}$. Suppose that (z_k) is a repelling or parabolic periodic orbit with essential orbit portrait $\mathcal{O}$. Then there exist ℓ and two periodic rays $g_{\underline{r}}$ and $g_{\underline{\tilde{r}}}$ landing at z_ℓ (the characteristic rays of the orbit (z_k)*) such that the curve $g_{\underline{r}} \cup \{z_\ell\} \cup g_{\underline{\tilde{r}}}$ separates the singular value from all other rays of the orbit portrait. The addresses $\underline{r}$ and $\underline{\tilde{r}}$ are called the* characteristic addresses *of $\mathcal{O}$; they depend on $\mathcal{O}$ but not on κ. The interval in $\mathcal{S}$ bounded by $\underline{r}$ and $\underline{\tilde{r}}$ is called the* characteristic sector *of $\mathcal{O}$.*

Furthermore, if there are at least three rays landing at each z_k, then the orbit portrait of (z_k) is of satellite type.

A pair $\langle \underline{r}, \underline{\tilde{r}} \rangle$ with $\underline{r} < \underline{\tilde{r}}$ is called a characteristic ray pair for E_κ *if E_κ has an orbit portrait whose characteristic rays are $\underline{r}$ and $\underline{\tilde{r}}$. More generally, $\langle \underline{r}, \underline{\tilde{r}} \rangle$ is called a* characteristic ray pair *if there exists some $\kappa \in \mathbb{C}$ with such an orbit portrait. If $\langle \underline{r}, \underline{\tilde{r}} \rangle$ is a characteristic ray pair of period n, then $\sigma^{n-1}(\underline{\tilde{r}}) < \sigma^{n-1}(\underline{r})$.* □

Remark. The final claim does not appear in the statement of [S2, Lemma 5.2], but is immediate from its proof.

3.4. Theorem and Definition (Characteristic Rays [S2, Lemma 5.2])**.**
Let κ be an attracting or parabolic parameter and let n be the length of $\underline{s} := \operatorname{addr}(\kappa)$. Then there exists a unique characteristic ray pair $\langle \underline{s}^-, \underline{s}^+ \rangle$ for E_κ such that the common landing point of $g_{\underline{s}^-}$ and $g_{\underline{s}^+}$ lies on the boundary of the characteristic Fatou component U_1; both rays have period n. This ray pair separates κ from all other periodic points with essential orbit portraits.

The addresses $\underline{s}^-$ and $\underline{s}^+$ depend only on $\underline{s}$, and are called the characteristic addresses *of $\underline{s}$ (or κ). The common landing point is called the* dynamic root *of E_κ.* □

Remark 1. In particular, the dynamic root is the unique boundary point of the characteristic Fatou component U_1 which is fixed under E_κ^n and which is the landing point of at least two dynamic rays.

Remark 2. We will later give an algorithm (Algorithm B.2) for determining $\langle \underline{s}^-, \underline{s}^+ \rangle$, given $\underline{s}$.

Remark 3. The case of parabolic parameters was not formally treated in [S2]. However, the proof is the same as that given there for attracting parameters. (Alternatively, the parabolic case follows from the attractive case by using Theorem 3.6 below.)

Itineraries and kneading sequences. Recall that, given an attracting or parabolic exponential map E_κ, one can connect the singular value to ∞ in the characteristic Fatou component U_1 by a curve γ at external address $\operatorname{addr}(\kappa)$. The preimage $E_\kappa^{-1}(\gamma)$ consists of

countably many curves in U_0, and these produce a partition of the dynamical plane. (The curve γ is unique up to homotopy within U_1, so the partition is natural except for points within U_0.) To any point $z \in J(E_\kappa)$, one can now associate an *itinerary*, which records through which strips of this partition the orbit of z passes. For more details, see [SZ2, Section 4].

In this article, we will use the following combinatorial analog of this notion. If $\underline{s} \in \mathcal{S}$, then $\sigma^{-1}(\underline{s})$ produces a partition of $\mathcal{S}$, and the itinerary of any $\underline{t} \in \mathcal{S}$ will record where the orbit of $\underline{t}$ under σ maps with respect to this partition.

3.5. Definition (Itineraries and Kneading Sequences)**.**
Let $\underline{s} \in \mathcal{S}$ and $\underline{r} \in \overline{\mathcal{S}}$. Then the itinerary of $\underline{r}$ with respect to $\underline{s}$ *is* $\operatorname{itin}_{\underline{s}}(\underline{r}) := \mathtt{u}_1\mathtt{u}_2\dots$, *where*

$$\begin{cases} \mathtt{u}_k = \mathtt{j} & \text{if} \quad \mathtt{j}\underline{s} < \sigma^{k-1}(\underline{r}) < (\mathtt{j}+1)\underline{s} \\ \mathtt{u}_k = {}^{\mathtt{j}}_{\mathtt{j}-1} & \text{if} \quad \sigma^{k-1}(\underline{r}) = \mathtt{j}\underline{s} \\ \mathtt{u}_k = * & \text{if} \quad \sigma^{k-1}(\underline{r}) = \infty. \end{cases}$$

(Note that $\operatorname{itin}_{\underline{s}}(\underline{r})$ *is is a finite sequence if and only if* $\underline{r}$ *is an intermediate external address.) We also define* $\operatorname{itin}^+_{\underline{s}}(\underline{r})$ *and* $\operatorname{itin}^-_{\underline{s}}(\underline{r})$ *to be the sequence obtained by replacing each boundary symbol* ${}^{\mathtt{j}}_{\mathtt{j}-1}$ *by* $\mathtt{j}$ *or* $\mathtt{j}-1$*, respectively. When κ is a fixed attracting or parabolic parameter, we usually abbreviate* $\operatorname{itin}(\underline{r}) := \operatorname{itin}_{\operatorname{addr}(\kappa)}(\underline{r})$.

We also define the kneading sequence *of $\underline{s}$ to be* $\mathbb{K}(\underline{s}) := \operatorname{itin}_{\underline{s}}(\underline{s})$*. Similarly, the upper and lower kneading sequences of $\underline{s}$ are* $\mathbb{K}^+(\underline{s}) := \operatorname{itin}^+_{\underline{s}}(\underline{s})$ *and* $\mathbb{K}^-(\underline{s}) := \operatorname{itin}^-_{\underline{s}}(\underline{s})$.

Remark 1. One should think of $\underline{s}$ as lying in the "combinatorial parameter plane", whereas $\underline{r}$ lies in the "combinatorial dynamical plane" associated with $\underline{s}$.

Remark 2. In the case $\underline{s} = \infty$, we can define itineraries analogously, but the addresses $\mathtt{j}\underline{s}$ and $(\mathtt{j}+1)\underline{s}$ in the definition will have to be replaced by $(\mathtt{j}-\frac{1}{2})\infty$ and $(\mathtt{j}+\frac{1}{2})\infty$. With this definition, $\operatorname{itin}_\infty(\underline{r}) = \underline{r}$ for all infinite external addresses $\underline{r}$.

Remark 3. The definition of itineraries involves a noncanonical choice of an offset for the labelling of the partition strips. Our choice was made so that the external addresses in the interval $(\overline{0}, \overline{1})$ are exactly those whose kneading sequences start with $\mathtt{0}$.

The significance of itineraries lies in the fact that they can be used to determine which periodic rays land together, as shown in [SZ2, Theorems 3.2 and 5.4, Proposition 4.5].

3.6. Theorem (Dynamic Rays and Itineraries)**.**
Let κ be an attracting or parabolic parameter. Then every periodic dynamic ray of E_κ lands at a periodic point, and conversely every repelling or parabolic periodic point is the landing point of such a ray.

Two periodic rays $g_{\underline{r}}$ and $g_{\underline{\tilde{r}}}$ land at the same point if and only if $\operatorname{itin}(\underline{r}) = \operatorname{itin}(\underline{\tilde{r}})$. □

The n-th itinerary entry $\mathtt{u}_n$ is locally constant (as a function of $\underline{r}$) wherever it is defined and an integer. If the n-th itinerary entry at $\underline{r}$ is a boundary symbol $\mathtt{u}_n = {}^{\mathtt{j}}_{\mathtt{j}-1}$, then it is

$\mathtt{j}-1$ slightly below $\underline{r}$ and $\mathtt{j}$ slightly above $\underline{r}$. In other words,

$$\lim_{\underline{t}\nearrow\underline{r}} \operatorname{itin}_{\underline{s}}(\underline{t}) = \operatorname{itin}^-_{\underline{s}}(\underline{r}) \qquad \text{and}$$
$$\lim_{\underline{t}\searrow\underline{r}} \operatorname{itin}_{\underline{s}}(\underline{t}) = \operatorname{itin}^+_{\underline{s}}(\underline{r})$$

for all infinite external addresses $\underline{r}$.

If the n-th itinerary entry of $\underline{r}$ is $*$, then $\underline{r}$ is an intermediate external address of length n. In this case, the n-th itinerary entries of addresses tend to $+\infty$ when approaching $\underline{r}$ from below and to $-\infty$ when approaching $\underline{r}$ from above.

We will frequently be in a situation where we compare the itineraries of an address $\underline{t}$ with respect to two different addresses $\underline{s}^1, \underline{s}^2 \in \mathcal{S}$. Therefore, let us state the following simple fact for further reference.

3.7. Observation (Itineraries and Change of Partition)**.**
Let $\underline{s}^1, \underline{s}^2, \underline{t} \in \mathcal{S}$ with $\underline{s}^1 < \underline{s}^2$ and let $j \geq 1$ such that $\sigma^j(\underline{t})$ is defined. Then the j-th entries of the itineraries of $\underline{t}$ with respect to $\underline{s}^1$ and $\underline{s}^2$ coincide if and only if $\sigma^j(\underline{t}) \notin [\underline{s}^1, \underline{s}^2]$.

In particular, $\operatorname{itin}^-_{\underline{s}^1}(\underline{t}) = \operatorname{itin}^+_{\underline{s}^2}(\underline{t})$ *if and only if $\sigma^k(\underline{t}) \notin (\underline{s}^1, \underline{s}^2)$ for all $k \geq 1$.*

Proof. By definition, the j-th itinerary entry of $\underline{t}$ as a function of $\underline{s} \in \mathcal{S}$ is locally constant on $\mathcal{S} \setminus \{\sigma^j(\underline{t})\}$, which proves the "if" part. On the other hand, this function jumps by 1 as $\underline{s}$ passes $\sigma^j(\underline{t})$, proving the "only if" part. ■

Finally, we shall require the following fact on the existence of addresses with prescribed itineraries.

3.8. Lemma (Existence of Itineraries)**.**
Let $\underline{s} \in \overline{\mathcal{S}}$. Let $\underline{\mathtt{u}}$ be either an infinite sequence of integers or a finite sequence of integers followed by $*$, and suppose that $\sigma^k(\underline{\mathtt{u}}) \notin \{\mathbb{K}^+(\underline{s}), \mathbb{K}^-(\underline{s})\}$ for all $k \geq 1$. Then there exists an external address $\underline{r} \in \overline{\mathcal{S}}$ with $\operatorname{itin}_{\underline{s}}(\underline{r}) = \underline{\mathtt{u}}$*; if $\underline{\mathtt{u}}$ is periodic then every such $\underline{r}$ is also periodic.*

Furthermore, if $\underline{t} \in \overline{\mathcal{S}}$ and $k \geq 0$, then no two elements of $\sigma^{-k}(\underline{t})$ have the same itinerary with respect to $\underline{s}$. (In particular, no two intermediate external addresses have the same itinerary with respect to $\underline{s}$.)

Remark. The condition $\sigma^k(\underline{\mathtt{u}}) \notin \{\mathbb{K}^+(\underline{s}), \mathbb{K}^-(\underline{s})\}$ is necessary. There exist periodic addresses $\underline{s}$ (for example $\underline{s} = \overline{0}$) such that both $\mathbb{K}^+(\underline{s})$ and $\mathbb{K}^-(\underline{s})$ are not realized as the itinerary of any external address. Similarly, as we will see in Lemma 7.6 (d), there exist nonperiodic addresses $\underline{s}$ with periodic kneading sequences. In this case, $\operatorname{itin}_{\underline{s}}(\underline{r}) \neq \mathbb{K}(\underline{s})$ for all periodic addresses $\underline{r}$.

Proof. The set R_k of all external addresses $\underline{r} \in \overline{\mathcal{S}}$ for which at least one of the itineraries $\operatorname{itin}^+_{\underline{s}}(\underline{r})$ and $\operatorname{itin}^-_{\underline{s}}(\underline{r})$ agrees with $\underline{\mathtt{u}}$ in the first k entries is easily seen to be compact and nonempty for all k. Thus $R := \bigcap_k R_k \neq \emptyset$. Let $\underline{r} \in R$; then $\underline{\mathtt{u}} \in \{\operatorname{itin}^+_{\underline{s}}(\underline{r}), \operatorname{itin}^-_{\underline{s}}(\underline{r})\}$.

We claim that $\operatorname{itin}_{\underline{s}}(\underline{r})$ contains no boundary symbols. Indeed, otherwise $\sigma^k(\underline{r}) = \underline{s}$ for some $k \geq 1$, and thus $\mathbb{K}(\underline{s}) = \sigma^k(\operatorname{itin}_{\underline{s}}(\underline{r}))$. Thus $\mathbb{K}^+(\underline{s}) = \sigma^k(\underline{\mathtt{u}})$ or $\mathbb{K}^-(\underline{s}) = \sigma^k(\underline{\mathtt{u}})$, which contradicts the assumption. Consequently, $\operatorname{itin}_{\underline{s}}(\underline{r}) = \operatorname{itin}^+_{\underline{s}}(\underline{r}) = \operatorname{itin}^-_{\underline{s}}(\underline{r}) = \underline{\mathtt{u}}$ as required.

The fact that $\underline{r}$ can be chosen to be periodic if $\underline{\mathtt{u}}$ is periodic is [SZ2, Lemma 5.2]. The proof that no aperiodic address can have the same itinerary as a periodic address is analogous to [M1, Lemma 18.12].

The final statement follows by induction from the trivial fact that changing the first entry of an address $\underline{t}$ by some integer m will also change the first entry of $\operatorname{itin}_{\underline{s}}(\underline{t})$ by m. ■

Properties of characteristic ray pairs. As a first application of the concept of itineraries, let us deduce two basic properties of characteristic ray pairs.

3.9. Lemma (Characteristic Ray Pairs)**.**
Let $\langle \underline{r}^-, \underline{r}^+\rangle$ be a characteristic ray pair of period n. Then there exist $\mathtt{u}_1, \dots \mathtt{u}_n \in \mathbb{Z}$ such that

$$\mathbb{K}^-(\underline{r}^-) = \mathbb{K}^+(\underline{r}^+) = \operatorname{itin}^-_{\underline{r}^-}(\underline{r}^+) = \operatorname{itin}^+_{\underline{r}^+}(\underline{r}^-) = \overline{\mathtt{u}_1 \dots \mathtt{u}_n}. \tag{2}$$

Furthermore, if $\underline{s} \in \mathcal{S}$, then the following are equivalent.

(a) $\underline{s} \in (\underline{r}^-, \underline{r}^+)$;
(b) $\operatorname{itin}_{\underline{s}}(\underline{r}^-) = \operatorname{itin}_{\underline{s}}(\underline{r}^+) = \overline{\mathtt{u}_1 \dots \mathtt{u}_n}$;
(c) $\operatorname{itin}_{\underline{s}}(\underline{r}^-) = \operatorname{itin}_{\underline{s}}(\underline{r}^+)$.

Proof. It follows from the definition of characteristic addresses that $\mathbb{K}^-(\underline{r}^-) = \operatorname{itin}^-_{\underline{r}^-}(\underline{r}^-) = \operatorname{itin}^-_{\underline{r}^-}(\underline{r}^+) = \overline{\mathtt{u}_1 \dots \mathtt{u}_n}$ for some $\mathtt{u}_1, \dots, \mathtt{u}_n \in \mathbb{Z}$. Furthermore, no forward image of $\underline{r}^-$ or $\underline{r}^+$ belongs to $(\underline{r}^-, \underline{r}^+)$. Thus Observation 3.7 implies that

$$\operatorname{itin}_{\underline{s}}(\underline{r}^-) = \operatorname{itin}_{\underline{s}}(\underline{r}^+) = \overline{\mathtt{u}_1 \dots \mathtt{u}_n}$$

for all $\underline{s} \in (\underline{r}^-, \underline{r}^+)$. This proves (2) as well as "(a)⇒(b)".

Clearly (b) implies (c). To prove that (a) follows from (c), let $\underline{s} \in \mathcal{S} \setminus (\underline{r}^-, \underline{r}^+)$. Since the interval $[\sigma^{n-1}(\underline{r}^+), \sigma^{n-1}(\underline{r}^-)]$ is mapped bijectively to $\overline{\mathcal{S}} \setminus (\underline{r}^-, \underline{r}^+)$ by the shift, there is an element of $\sigma^{-1}(\underline{s})$ between $\sigma^{n-1}(\underline{r}^+)$ and $\sigma^{n-1}(\underline{r}^-)$. So the n-th itinerary entries of $\underline{r}^-$ and $\underline{r}^+$ with respect to $\underline{s}$ are different, as required. ■

3.10. Lemma (Unique Intermediate Addresses)**.**
Let $\langle \underline{r}^-, \underline{r}^+\rangle$ be a characteristic ray pair of period n, and let $\mathtt{u}_1, \dots \mathtt{u}_n \in \mathbb{Z}$ be as in the previous lemma. Moreover, let $\underline{s}$ be any intermediate external address of length n. Then the following are equivalent:

(a) *$\underline{r}^-$ and $\underline{r}^+$ are the characteristic addresses of $\underline{s}$;*
(b) $\operatorname{itin}_{\underline{r}^-}(\underline{s}) = \operatorname{itin}_{\underline{r}^+}(\underline{s}) = \mathbb{K}(\underline{s}) = \mathtt{u}_1 \dots \mathtt{u}_{n-1}*$;
(c) $\underline{s} \in (\underline{r}^-, \underline{r}^+)$ *and* $\mathbb{K}(\underline{s}) = \mathtt{u}_1 \dots \mathtt{u}_{n-1}*$.

Furthermore, there is at most one $\underline{s}$ with one (and thus all) of these properties.

Remark. Note that we do not claim here that such an address $\underline{s}$ always exists. While it is not very difficult to show that this is indeed the case, we will not require this fact until Section 7, and it will be proved there (Proposition 7.4).

Proof. If $\underline{r}$ and $\underline{r}^+$ are the characteristic adresses of $\underline{s}$, then no forward image of $\underline{s}$ lies in $(\underline{r}^-, \underline{r}^+)$. Indeed, let κ be a parameter with $\operatorname{addr}(\kappa) = \underline{s}$. Then the characteristic rays $g_{\underline{r}^-}$ and $g_{\underline{r}^+}$ separate the characteristic Fatou component U_1 from the other periodic Fatou components, and thus from the remainder of the singular orbit. So (a) implies (b) by Observation 3.7.

Suppose that (b) holds. Then, for each $k \in \{1, \dots, n-1\}$, the three addresses $\sigma^{k-1}(\underline{r}^-)$, $\sigma^{k-1}(\underline{r}^+)$ and $\sigma^{k-1}(\underline{s})$ belong to the interval $(\mathtt{u}_k\underline{r}^-, (\mathtt{u}_k+1)\underline{r}^-]$. By Observation 2.2, the map σ^{n-1} thus preserves the circular order of $\underline{r}^-$, $\underline{r}^+$ and $\underline{s}$. Recall that $\sigma^{n-1}(\underline{r}^+) < \sigma^{n-1}(\underline{r}^-)$ by Lemma 3.3. Since $\sigma^{n-1}(\underline{s}) = \infty$, it follows that $\underline{r}^- < \underline{s} < \underline{r}^+$. By Lemma 3.9, it follows that (c) holds.

Now let us assume that $\underline{s}$ satisfies (c), and let $\kappa \in \mathbb{C}$ with $\operatorname{addr}(\kappa) = \underline{s}$. Then the dynamic rays $g_{\underline{r}^-}$ and $g_{\underline{r}^+}$ have a common landing point w by Lemma 3.9 and Theorem 3.6. It easily follows that $\underline{r}^-$ and $\underline{r}^+$ are the characteristic addresses of the orbit portrait of w.

Now let $\underline{s}^-$ and $\underline{s}^+$ be the characteristic addresses of $\underline{s}$. Then we have

$$\underline{r}^- \leq \underline{s}^- < \underline{s} < \underline{s}^+ \leq \underline{r}^+,$$

and the first $n-1$ itinerary entries of all these addresses with respect to $\underline{s}$ coincide. As above, the cyclic order of this configuration is preserved under σ^{n-1}. Since $\sigma^{n-1}(\underline{s}) = \infty$, this means that

$$\sigma^{n-1}(\underline{s}^+) \leq \sigma^{n-1}(\underline{r}^+) < \sigma^{n-1}(\underline{r}^-) \leq \sigma^{n-1}(\underline{s}^-). \tag{3}$$

Since the outer two addresses in (3) have the same itinerary, this means that the n-th itinerary entries of $\underline{s}^-, \underline{s}^+, \underline{r}^-$ and $\underline{r}^+$ also agree. Thus all these addresses belong to the orbit portrait of w. Since both pairs $\langle \underline{s}^-, \underline{s}^+ \rangle$ and $\langle \underline{r}^-, \underline{r}^+ \rangle$ are characteristic ray pairs of this portrait, they must be equal. This proves (a).

Finally, by Lemma 3.8, there is at most one $\underline{s} \in \mathcal{S}$ with $\operatorname{itin}_{\underline{r}^-}(\underline{s}) = \mathtt{u}_1 \dots \mathtt{u}_{n-1}*$, which completes the proof. ∎

4. Hyperbolic components

A *hyperbolic component* W is a maximal connected open subset of parameter space in which each parameter has an attracting periodic orbit. It is easy to see that this (unique) orbit depends holomorphically on κ, and in particular its period is constant throughout W.

It was shown in [BR] that every hyperbolic component W is simply connected and unbounded. Furthermore, the multiplier map $\mu : W \to \mathbb{D}^*$, which maps every parameter to the multiplier of its attracting cycle, is a universal covering. Since $\exp : \mathbb{H} \to \mathbb{D}^*$ is also a universal covering, there exists a conformal isomorphism $\Psi_W : \mathbb{H} \to W$ with $\mu \circ \Psi_W = \exp$. Note that this defines Ψ_W uniquely only up to precomposition by a deck transformation

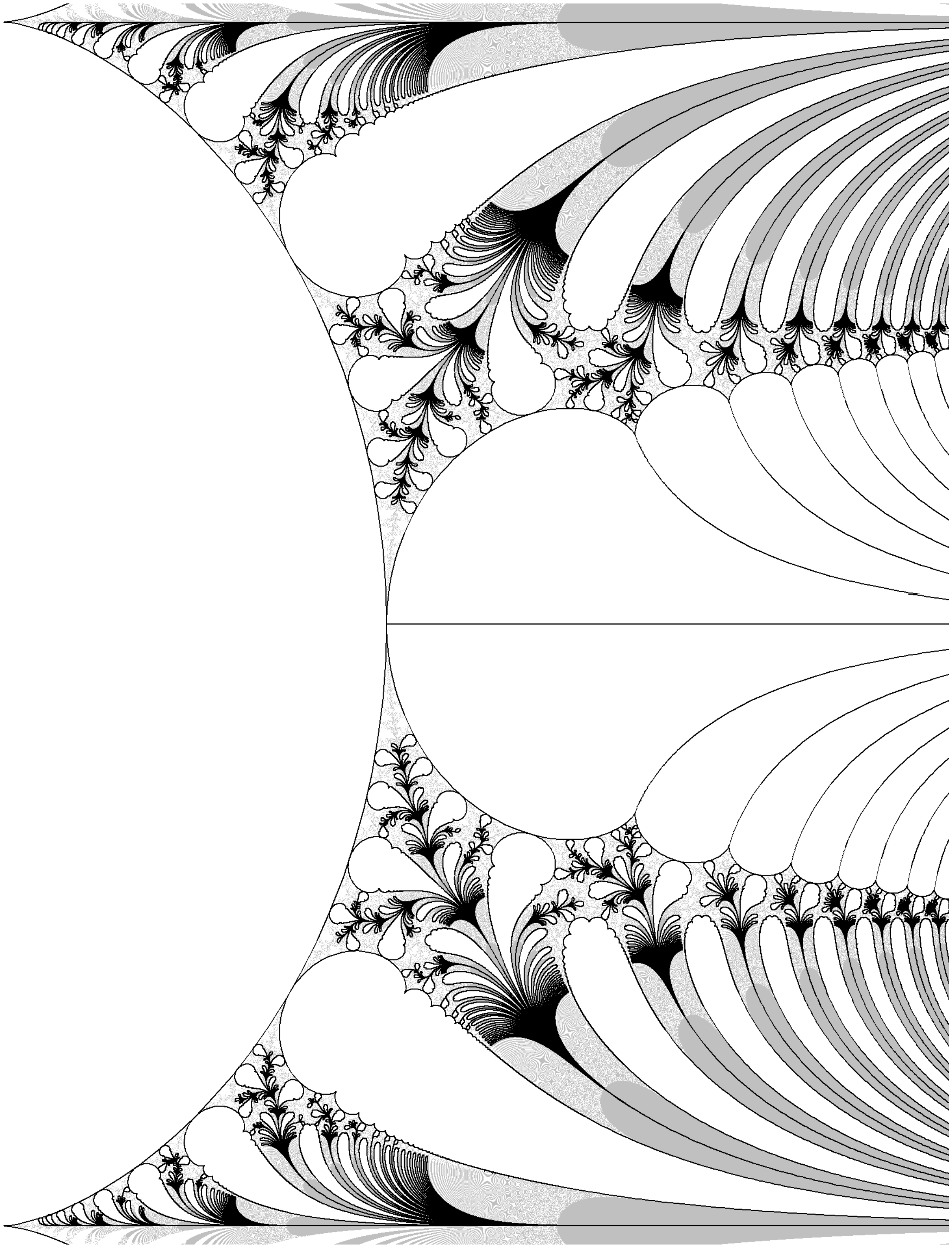


FIGURE 3. Several hyperbolic components in the strip $\{\operatorname{Im} \kappa \in [0, 2\pi]\}$. Within the period two component in the center right of the picture, parameter rays at integer heights are drawn in.

of exp (i.e., an additive translation by $2\pi ik$, $k \in \mathbb{Z}$). However, in [S2, Theorem 7.1], it was shown that, when W has period at least two, there is a unique choice of Ψ_W such that, for any $\kappa \in \Psi_W\big((-\infty, 0)\big)$, the dynamic root lies on the distinguished boundary orbit. (This also follows from Theorem 4.4 below.) In the following, we will always fix Ψ_W to be this *preferred parametrization.*

It is well-known [BR] that there exists a unique hyperbolic component W_0 of period 1. This hyperbolic component contains a left half plane and is invariant under $z \mapsto z + 2\pi i$. Since different choices for the parametrization of this component only correspond to such a translation, and thus to a relabeling of the map, there is no canonical choice for a preferred parametrization. By definition, we choose the *preferred parametrization* of the period 1 component to be the unique map respecting the real symmetry, i.e.

$$\Psi_W : \mathbb{H} \to W_0; z \mapsto z - \exp(z).$$

Internal rays. We can now foliate W by curves, called *internal rays*, along which the argument of μ is constant. More precisely, the *internal ray at height* $h \in \mathbb{R}$ is the curve

$$\Gamma_{W,h} : (-\infty, 0) \to \mathbb{C}, t \mapsto \Psi_W(t + 2\pi ih).$$

It is straightforward to see that, if W is a hyperbolic component of period ≥ 2, then $\Gamma_{W,h}(t) \to +\infty$ as $t \to \infty$, and the homotopy class of this curve as $t \to \infty$ is independent of h (see e.g. [S2, Lemma 2.1]). We say that an internal ray $\Gamma_{W,h}$ *lands* at a point $\kappa \in \hat{\mathbb{C}}$ if $\kappa = \lim_{t \nearrow 0} \Gamma_{W,h}(t)$.

4.1. Lemma (Landing of Internal Rays)**.**
Let W be a hyperbolic component of period n. Then every internal ray $\Gamma_{W,h}$ lands at some point in $\hat{\mathbb{C}}$, which we denote by $\Psi_W(2\pi ih)$. The set of h for which this landing point belongs to $\mathbb{C}$ is open and dense. Conversely, suppose that $\kappa_0 \in \partial W \cap \mathbb{C}$. Then κ_0 is the landing point of a unique internal ray $\Gamma_{W,h}$. Furthermore, κ_0 is an indifferent parameter of period dividing n. If a is a point on this indifferent periodic orbit, then $(E^n_{\kappa_0})'(a) = \exp(2\pi ih)$.

Remark 1. It is not difficult to see that the extended map Ψ_W is continuous on $\overline{\mathbb{H}}$. However, we shall not require this result here.

Remark 2. This lemma leaves open the possibility that some internal rays land at ∞, disconnecting the boundary of $\mathbb{C}$. Proving that this does not happen is much more difficult. The proof of this fact, in [RS], uses the results of the present article. (An outline of the argument was given in [S3].) This result can therefore not be used in the following sections.

Sketch of proof. It is straightforward to see that every point of $\partial W \cap \mathbb{C}$ has an indifferent periodic orbit of period dividing n. The multiplier map μ extends to a holomorphic function on a neighborhood of κ_0 (or on a finite-sheeted covering of κ_0 when $\mu(\kappa_0) = 1$, compare [M3, Proof of Lemma 4.2]). Hence there exists some internal ray which lands at κ_0. If there were two internal rays of W landing at κ_0, these could be connected to

form a simple closed curve $\gamma \subset \overline{W}$ which separates some part of ∂W from infinity. It is a standard fact that this is not possible: for example, it is well-known that every point of $\partial W \cap \mathbb{C}$ is structurally unstable, and thus can be approximated by attracting parameters with arbitrarily high periods (compare [R1, Lemma 5.1.6]). The hyperbolic components containing these parameters thus will be separated from ∞ by γ, which is impossible since every hyperbolic component is unbounded.

The fact that the set of h for which $\Gamma_{W,h}(0)$ lands in $\mathbb{C}$ is open follows easily from the above statement about the multiplier. That this set is also dense (and in fact has full measure) follows from the F. and M. Riesz theorem [M1, Theorem A.3].

Finally, it is straightforward to see that every finite limit point κ_0 of an internal ray $\Gamma_{W,h}$ has a periodic point a with $(E^n_{\kappa_0})'(a) = \exp(2\pi i h)$. The set of such parameters is easily seen to be discrete in $\mathbb{C}$, proving that $\Gamma_{W,h}$ lands at a point of $\hat{\mathbb{C}}$.

For a more detailed (self-contained!) proof of this lemma, compare [RS, Section 2]. ■

When W is of period at least 2, the internal ray at height 0 is called the *central internal ray* of W. If this ray lands at a point in $\mathbb{C}$, then its landing point $\Psi_W(0)$ is called the *root* of W. The points of $\Psi_W(2\pi i\mathbb{Z} \setminus \{0\}) \cap \mathbb{C}$ are called *co-roots* of W. For the period 1 component, all points of $\Psi_W(2\pi i\mathbb{Z})$ are called co-roots. Recall that we will prove in [RS] that every component has a root (and infinitely many co-roots). Without this knowledge, which we may not use at this point, we cannot be sure that bifurcations actually exist.

Classification of hyperbolic components. It is easy to see that $\operatorname{addr}(\kappa)$ depends only on the hyperbolic component W which contains κ; this address will therefore also be denoted by $\operatorname{addr}(W)$. Similarly, we will talk about the kneading sequence, characteristic rays etc. of W. The following theorem, which is the main result of [S2], states that hyperbolic components can be completely classified terms of their combinatorics. (Note that the existence part of this result was already cited as Theorem 2.3.)

4.2. Proposition (Classification of Hyperbolic Components [S2]).
For every intermediate external address $\underline{s}$, there exists exactly one hyperbolic component W with $\operatorname{addr}(W) = \underline{s}$. We denote this component by $\operatorname{Hyp}(\underline{s})$. The vertical order of hyperbolic components coincides with the lexicographic order of their external addresses. □

To explain the last statement, recall that, when W is a hyperbolic component of period ≥ 2, any internal ray $\Gamma_{W,h}$ satisfies $\Gamma_{W,h}(t) \to +\infty$ as $t \to -\infty$. Thus the family of central internal rays of hyperbolic components has a natural vertical order[2] as described in Section 2, and this is the order referred to in the Proposition. (Note that taking the central rays is not essential, as there is only one homotopy class of curves in W along which the multiplier tends to 0.)

[2]We should stress that we only use the "negative" ends of internal rays to define this order. A priori some internal rays might also tend to infinity as $t \to +\infty$. Taking these directions would result in a different order which we do not refer to.

Sectors. Since $\mu : W \to \mathbb{D}^*$ is a universal covering, parameters in W are not (as in the quadratic family) uniquely determined by their multiplier. Rather, the set $\mu^{-1}(\mathbb{D}^* \setminus [0,1))$ consists of countably many components, called *sectors* of W. If $\kappa = \Psi_W(t + 2\pi i h)$ with $h \notin \mathbb{Z}$, we denote the sector containing κ by

$$\mathrm{Sec}(\kappa) := \mathrm{Sec}(W,h) := \Psi_W\Big(\big\{a + 2\pi i b : a < 0 \text{ and } b \in (\lfloor h \rfloor, \lceil h \rceil)\big\}\Big).$$

4.3. Definition (Sector Labels)**.**
Let $W = \mathrm{Hyp}(\underline{s})$ be a hyperbolic component and let $\kappa \in W$ be a parameter with $\mu := \mu(\kappa) \notin (0,1)$. Let γ be the principal attracting ray of E_κ, and let γ' be the component of $E_\kappa^{-1}(\gamma)$ which starts at a_0. Then $\mathrm{addr}(\gamma')$ is of the form $s_*\underline{s}$ with $s_* \in \mathbb{Z}$ (resp. $s_* \in \mathbb{Z} + \frac{1}{2}$ if $\underline{s} = \infty$). The entry $s_* = s_*(\kappa)$ is called the sector label *of κ.*

Remark. There are two more ways to label sectors which will appear in this article: *sector numbers* and *kneading entries*; both will be introduced in Section 7. We should warn the reader that our terminology is somewhat different from that of [LS], where the term *sector label* is used to refer to what we call *kneading entries*.

The following results justify the term "sector label"; compare Figure 5(a).

4.4. Theorem (Behavior of Sector Labels)**.**
The map $\kappa \mapsto s_*(\kappa)$ is constant on sectors of W. When κ crosses a sector boundary so that μ passes through $(0,1)$ in positive orientation, then $s_*(\kappa)$ increases exactly by 1. In particular the induced map from sectors to indices is bijective. The unique sector with a given sector label s_* will be denoted by $\mathrm{Sec}(W, s_*)$.

If the period of W is at least 2 and κ is a parameter on the internal ray between $\mathrm{Sec}(W,j)$ and $\mathrm{Sec}(W,j+1)$, then the distinguished boundary fixed point of E_κ^n on ∂U_1 has itinerary $\overline{\mathtt{u}_1 \dots \mathtt{u}_{n-1} j}$ (where $\mathbb{K}(W) = \mathtt{u}_1 \dots \mathtt{u}_{n-1} *$). In particular, the central internal ray of W is the boundary between $\mathrm{Sec}(W,j)$ and $\mathrm{Sec}(W,j+1)$, where j and $j+1$ are the n-th entries of the characteristic addresses $\underline{s}^+$ and $\underline{s}^-$, respectively.

If W is the unique period one component, a similar statement holds: the boundary between $\mathrm{Sec}(W, j-\frac{1}{2})$ and $\mathrm{Sec}(W, j+\frac{1}{2})$ is given by the internal ray $\{t + 2\pi i j : t \leq -1\}$. For parameters on this ray, the distinguished boundary fixed point of E_κ has itinerary $\overline{j}$.

Sketch of proof. The linearizing coordinate used to define attracting dynamic rays depends holomorphically on κ. It easily follows that, as long as the principal attracting ray γ does not pass through κ, its preimage $\gamma' = \gamma'(\kappa)$ from Definition 4.3 varies continuously, which shows that s_* is constant on sectors.

In the following, let us restrict to the case where the period of W is at least two; the case of the period one component is handled analogously. Let κ_0 be a parameter with positive real multiplier, and let us set $\underline{s} := \mathrm{addr}(W)$. Then the principal attracting ray γ contains the singular value. Denote the piece of γ which connects a_1 to κ by γ_0 and the piece which connects κ to $+\infty$ by γ_1. We can define a branch φ of E_κ^{-1} on $U_1 \setminus \gamma_1$ which takes a_1 to

a_0. The range of φ is then a strip S of U_0 bounded by two consecutive preimages ${\gamma'}_1^-$ and ${\gamma'}_1^+$, at external addresses $j\underline{s}$ and $(j+1)\underline{s}$ for some $j \in \mathbb{Z}$.

Let $\alpha \subset U_1 \backslash \gamma_1$ be an unbroken attracting dynamic ray, connecting a_1 to the distinguished boundary fixed point $w \in \partial U_1$. Then the image of α under φ is a curve connecting a_0 to $E_\kappa^{n-1}(w)$. Thus E_κ^{n-1} lies between ${\gamma'}_1^-$ and ${\gamma'}_1^+$, so j is the n-th itinerary entry of w as claimed.

Denote the preimage of γ_0 in S by γ_0'. We then define two curves (in $\hat{\mathbb{C}}$),

$$\gamma'^{\pm} := \gamma_0' \cup \{-\infty\} \cup {\gamma'}_1^{\pm}.$$

By continuity of the linearizing coordinate, it then follows that, as $\kappa \to \kappa_0$ through parameters at positive (resp. negative) multiplier angles, the curves $\gamma'(\kappa)$ converge uniformly to γ'^+ (resp. γ'^-), which completes the proof. (Compare [R1, Theorem 5.5.3] for more details.) $\square$

5. Local Bifurcation Results

Throughout this section, let κ_0 be a parabolic parameter of period n and intermediate external address $\underline{s}$. If the parabolic orbit portrait of E_{κ_0} is essential, then we call κ_0 a *satellite* or a *primitive* parameter, depending on the type of this orbit portrait. Similarly, we will refer to the ray period of this orbit portrait as the ray period of κ_0. Note that this ray period is also the period of the repelling (or attracting) petals of the parabolic orbit.

In this section, we will study what happens when κ_0 is perturbed into an adjacent hyperbolic component. For this purpose, we will use the following well-known statement about the analytic structure near κ_0. This result is beautifully exposed, and proved using elementary complex analysis, in [M3, Section 4]. All that this local analysis requires is that there is only one singular value, and thus only one single cycle of petals at a parabolic periodic point.

5.1. Proposition (Perturbation of Parabolic Orbits)**.**
Let κ_0 be a parabolic parameter of period n, with ray period qn.

- ***(Primitive and Co-root case)*** *If $q = 1$ (so the multiplier of the parabolic orbit is 1), then, under perturbation, the parabolic orbit splits up into two orbits of period n that can be defined as holomorphic functions of a two-sheeted cover around κ_0.*

  Any hyperbolic component whose boundary contains κ_0 corresponds to one of these orbits becoming attracting (and therefore has period n).
- ***(Satellite Case)*** *If $q \geq 2$, then, under perturbation, the parabolic orbit splits into one orbit of period n and one of period qn. The period n orbit can be defined as a holomorphic function in a neighborhood of κ_0, as can the multiplier of the period qn-orbit. The qn-orbit itself can be defined on a q-sheeted covering around κ_0.*

  Any hyperbolic component whose boundary contains κ_0 corresponds to one of these orbits becoming attracting (and therefore has period n or qn). $\square$

Any hyperbolic component of period qn that touches κ_0 is called a *child component*; note that at least one such component always exists. In the satellite case, any period n

component touching κ_0 is called a *parent component*. Note that any satellite parameter has at least one child and at least one parent component; a primitive parameter has at least one child component but no parent components. (We will show in Theorems 5.3 and 5.4 below that "at least one" can be replaced by "exactly one".)

We will also require the following statement on the landing behavior of periodic rays as κ_0 is perturbed. The proof is analogous to that in the case of quadratic polynomials, which can also be found in [M3, Section 4]. (Recall that, by the previous proposition, the parabolic orbit of κ_0 breaks up into two orbits under perturbation. If we perturb κ_0 into an adjacent hyperbolic component, one of these orbits becomes attracting, so there is a unique repelling orbit created in the bifurcation.)

5.2. Proposition (Orbit Stability under Perturbation)**.**
Under perturbation of a parabolic parameter κ_0 into a child or parent component, all repelling periodic points retain the same orbit portraits.

Furthermore, under perturbation into a child component, the repelling periodic orbit created in the bifurcation has the same orbit portrait as the parabolic orbit of E_{κ_0}. Under perturbation into a parent component, the rays landing at the parabolic orbit are split up, landing at distinct points of the newly created repelling orbit. □

We are now ready to describe the combinatorics of child and parent components of κ_0 (and, in particular, show that there is at most one of each, see Corollary 5.5.

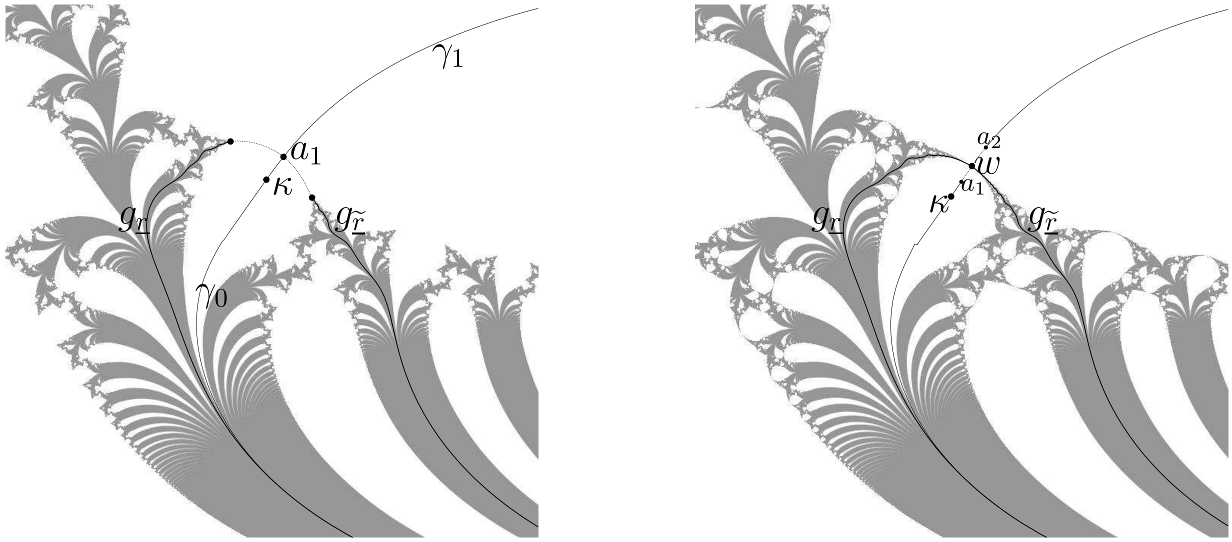
5.3. Theorem (Combinatorics in a Child Component)**.**
Let κ_0 be a parabolic parameter of period n and ray period qn, and let W be a child component of κ_0. Then $\operatorname{addr}(W) = \operatorname{addr}(\kappa_0)$*; i.e., W is the unique component at address* $\operatorname{addr}(\kappa_0)$.

Furthermore, for points on the internal ray of W landing at κ_0, the repelling point created in the bifurcation is the distinguished boundary fixed point. Therefore κ_0 is the root point of W if and only if its parabolic orbit portrait is essential; otherwise, κ_0 is a co-root of W.

Proof. If $qn = 1$, then $W = \operatorname{Hyp}(\infty)$ is the unique component of period 1. Now suppose that $qn > 1$. Let $\Gamma_{W,h}$ be the unique internal ray landing at κ_0 (compare Lemma 4.1); this ray has integer height $h \in \mathbb{Z}$. Let $\kappa := \Gamma_{W,h}(t)$ be a parameter on this ray. By Proposition 5.2, κ_0 and κ have the same orbit portraits and thus they have the same characteristic addresses. Lemma 3.10 then yields $\operatorname{addr}(\kappa_0) = \operatorname{addr}(\kappa) = \operatorname{addr}(W)$.

Now let w be the newly created repelling point and let w' be the distinguished boundary fixed point of κ. Let α be the piece of the principal attracting ray of E_κ which connects κ to ∞. Recall from Lemma 2.4 that a_0 and $E_\kappa^{n-1}(w')$ can be connected by an unbroken dynamic ray of E_κ and thus belong to the same component of $\mathbb{C} \setminus E_\kappa^{-1}(\alpha)$. We will show that a_0 and $E_\kappa^{n-1}(w)$ are also not separated by $E_\kappa^{-1}(\alpha)$. This implies that w and w' have the same itinerary and are therefore equal by Theorem 3.6, as required.

Let $\Phi : U_1 \to \mathbb{C}$ be the linearizing coordinate for E_κ, normalized so that $\Phi(\kappa) = 1$, and let $V \subset U_1$ be the component of the preimage of $\Phi^{-1}(\mathbb{D}(0, \frac{1}{\mu}))$ which contains a_1.



(a) parent component

(b) child component

FIGURE 4. The dynamical plane just before and after a bifurcation, illustrating Theorems 5.3 and 5.4. In both pictures, unbroken attracting rays are dotted lines, broken attracting rays are solid lines, and dynamic rays are strong solid lines.

(Note that $\mathbb{D}(0, \frac{1}{\mu})$ is the largest disk on which Φ^{-1} exists, and that $\mu = \exp(t)$.) Then V contains the curve $\gamma := E_\kappa^n(\alpha)$, and by definition $\Phi(\gamma) = (\mu, 1)$. Since $\Phi|_V : V \to \mathbb{D}(0, \frac{1}{\mu})$ is a conformal isomorphism, the hyperbolic length of γ in V is

$$\ell_V(\gamma) = \int_\mu^1 \frac{\mu|dz|}{1 - (\mu|z|)^2} \leq \frac{\mu(1-\mu)}{1-\mu^2} = \frac{\mu}{1+\mu} \leq 2.$$

In other words, the hyperbolic length of γ within U_1 stays bounded as $t \to 0$. Since the euclidean length of γ (which connects κ and $E_\kappa^n(\kappa)$) is bounded below as $t \to 0$, it follows from standard estimates on the hyperbolic metric, using the fact that $w \in \partial U_1$, that the euclidean distance $\operatorname{dist}(w, \gamma)$ is also bounded below as $t \to 0$.

Since E_κ^{n+1} is continuous in z and κ,

$$\liminf_{t \to 0} \operatorname{dist}(E_\kappa^{n-1}(w), E_\kappa^{-1}(\alpha)) > 0$$

(recall that $E_\kappa^n(w) = w$). On the other hand, the attracting point a_0 and the repelling point $E_\kappa^{n-1}(w)$ are created from the same parabolic point, so the distance between them tends to 0 as $t \to 0$. Thus, for small enough t, these two points are not separated by $E_\kappa^{-1}(\alpha)$. Therefore we have shown $w = w'$.

To prove the final statement, observe that by Proposition 5.2, the parabolic orbit portrait of E_{κ_0} is essential if and only if the orbit portrait of $w = w'$ is essential for E_κ. By definition,

the latter is the case if and only if κ belongs to the central internal ray; i.e., if κ_0 is the root point of W. ■

Let us now turn to determining the parent component of a satellite parameter.

5.4. Theorem (Combinatorics in a Parent Component)**.**
Let κ_0 be a satellite parabolic parameter of period n and multiplier $e^{2\pi i \frac{p}{q}}$; let W be a parent component of κ_0. If κ is a parameter on the internal ray in W which lands at κ_0, then the distinguished boundary orbit for κ is the repelling period qn orbit created from the parabolic point.

Furthermore, for such a parameter κ, let $\underline{s}'$ be the address of the attracting dynamic ray which contains the singular value. Then $\operatorname{addr}(\kappa_0) = \underline{s}'$.

Proof. To prove the first statement, choose some small wedge-shaped repelling petals for the parabolic orbit of E_{κ_0}. If κ is close enough to κ_0 on the internal ray landing at κ_0, then, for κ, these petals (after a translation moving them to the attracting period n orbit (a_k) created in the bifurcation) are still backward invariant and contain the newly created repelling period qn orbit (w_k) (see [M3, Section 4]). Choose any attracting dynamic ray which approaches a_1 through one of these petals. Recall that this ray is periodic under E_κ^n, of period q. Pulling back, it must land at the unique fixed point of the return map of this petal. That is, the points of the repelling orbit (w_k) are landing points of a cycle of attracting dynamic rays; thus (w_k) is the distinguished boundary orbit.

To prove the second part of the theorem, let $\underline{r}, \underline{\tilde{r}}$ be the characteristic addresses of κ_0. We will now find the combinatorial features of κ_0 within the attracting dynamics of κ, using attracting dynamic rays (compare Figure 4(b).) Let γ_0 be the attracting dynamic ray containing the singular value (recall that $\underline{s}' = \operatorname{addr}(\gamma_0)$). Then the attracting rays $\gamma_j := E_\kappa^{nj}(\gamma_0)$, $j = 0, \dots, q-1$ completely contain the singular orbit in U_1. By Proposition 5.2 and the first part of the theorem, the rays $g_{\underline{r}}$ and $g_{\underline{\tilde{r}}}$ do not land together for E_κ, but rather land separately on two points of the distinguished boundary cycle on ∂U_1. These landing points can be connected to a_1 by two attracting dynamic rays. Let $h_{\underline{r}}$ and $h_{\underline{\tilde{r}}}$ denote the curves obtained by extending $g_{\underline{r}}$ and $g_{\underline{\tilde{r}}}$ to a_1 by these attracting rays.

Now consider the preimages of that part of γ_0 which connects κ to ∞; these curves are straight lines in the linearizing coordinate of U_0, and connect $-\infty$ to $+\infty$ with external addresses of the form $m\underline{s}'$. The images of $h_{\underline{r}} \cup \gamma_0 \cup h_{\underline{\tilde{r}}}$ do not intersect these curves. Consequently for every $j \geq 0$, the j-th iterated images of $g_{\underline{r}}$, $g_{\underline{\tilde{r}}}$ and γ_0 belong to a common strip of this partition.

Thus, $\operatorname{itin}_{\underline{s}'}(\underline{r}) = \operatorname{itin}_{\underline{s}'}(\underline{\tilde{r}})$; so in particular $\underline{s} \in (\underline{r}, \underline{\tilde{r}})$ by Lemma 3.9. Furthermore, this itinerary agrees with $\mathbb{K}(\underline{s}')$ on its first $qn-1$ entries, which implies by Lemma 3.10 that $\underline{r}$ and $\underline{\tilde{r}}$ are the characteristic addresses of $\underline{s}'$. Since $\underline{r}$ and $\underline{\tilde{r}}$ are also the characteristic addresses of κ_0, we conclude that $\underline{s}' = \operatorname{addr}(\kappa_0)$. ■

5.5. Corollary (Bifurcation Structure at a Parabolic Point)**.**
Suppose that κ_0 is a parabolic parameter of period n and ray period qn. Let $\underline{s} := \operatorname{addr}(\kappa)$ and $\mathbb{K}(\underline{s}) = \mathtt{u}_1 \dots \mathtt{u}_{qn-1}*$. Then $\partial \operatorname{Hyp}(\underline{s})$ is the unique child component of κ_0; if κ_0 is

of satellite type, then $\mathrm{Hyp}(\sigma^{(q-1)n}(\underline{s}))$ *is the unique parent component of* κ_0*. No other hyperbolic component contains* κ_0 *on its boundary.*

Furthermore, κ_0 *is the root point of* $\mathrm{Hyp}(\underline{s})$ *if and only if the parabolic orbit of* E_{κ_0} *has essential orbit portrait (which is always the case if* $q > 1$*). Otherwise,* κ_0 *is a co-root of* $\mathrm{Hyp}(\underline{s})$*.*

Proof. Let W be a child component of κ_0 (at least one such component always exists). By Theorem 5.3, $\mathrm{addr}(W) = \underline{s}$. By the classification of hyperbolic components (Proposition 4.2), this component is unique. Similarly, suppose that $q > 1$; i.e., that κ_0 is a satellite parameter. By Theorem 5.4, $\mathrm{Hyp}(\sigma^{(q-1)n}(\underline{s}))$ is the unique parent component of κ_0. By Proposition 5.1, no other hyperbolic component contains κ on its boundary. The final statement was already proved in Theorem 5.3. ■

6. Bifurcation from a Hyperbolic Component

Suppose that κ_0 is a satellite parabolic parameter with parent component W and child component V. It follows from Theorem 5.4 that the address of the child component V is determined by W and the height $h \in \mathbb{Q} \setminus \mathbb{Z}$ of the internal ray $\Gamma_{W,h}$ landing at κ_0: it is the intermediate external address of a curve situated in the dynamical plane of $\kappa \in \Gamma_{W,h}$. However, this address is defined for *every* $h \in \mathbb{Q} \setminus \mathbb{Z}$, regardless of whether we know that the corresponding internal ray has a landing point!

We now use this idea to prove a combinatorial analog of the description of the structure of components bifurcating from a hyperbolic component W, as outlined in the introduction. More precisely, for every $h \in \mathbb{Q} \setminus \mathbb{Z}$, we define the address $\mathrm{addr}(W, h)$ of the hyperbolic component which "would" bifurcate from W at height h if the corresponding bifurcation parameter existed. Since these addresses are naturally defined in terms of curves in the dynamical plane of E_κ for $\kappa \in W$, it is straightforward to compute their itineraries with respect to $\underline{s} := \mathrm{addr}(W)$. Using this information, we can show that $\mathrm{addr}(W, h)$ behaves as we expect, working exclusively within the combinatorial dynamical plane associated to $\underline{s}$.

Combinatorial Bifurcation. As discussed above, Theorem 5.4 suggests the following definition.

6.1. Definition (Combinatorial Bifurcation)**.**
Let $W = \mathrm{Hyp}(\underline{s})$ *be a hyperbolic component of period* n*, and let* κ *be a parameter on the internal ray* $\Gamma_{W,h}$ *for some* $h \in \mathbb{Q} \setminus \mathbb{Z}$*. Let* $\underline{s}'$ *be the external address of the attracting dynamic ray of* E_κ *which contains the singular value. Then we say that* $\mathrm{Hyp}(\underline{s}')$ bifurcates combinatorially *from* W *(at height* h*). We denote the address of this component by*

$$\mathrm{addr}(W, h) := \mathrm{addr}(A, p/q) := \mathrm{addr}(\underline{s}, s_*, p/q) := \underline{s}'$$

where $A := \mathrm{Sec}(W, s_*)$ *is the sector containing* κ *and* $p/q = h - \lfloor h \rfloor$ *is the fractional part of* h*. The component* $\mathrm{Hyp}(\underline{s}')$ *is called a* (combinatorial) child component *of* W *and denoted by* $\mathrm{Bif}(W, h)$*.*

6.2. Proposition (Child Components)**.**
Let W be a hyperbolic component of period n, and let $h \in \mathbb{Q} \setminus \mathbb{Z}$. Then the following hold.

(a) $\kappa_0 := \Psi_W(2\pi i h) = \Psi_{\mathrm{Bif}(W,h)}(0) \quad (\in \hat{\mathbb{C}})$.
(b) *If $\kappa_0 \in \mathbb{C}$, then W and* $\mathrm{Bif}(W,h)$ *are the only hyperbolic components containing κ_0 on their boundaries.*
(c) *If two hyperbolic components have a common parabolic boundary point, then one of these components is a child component of the other.*
(d) *If $\kappa \in \Gamma_{W,h}$, then the dynamic rays of E_κ at the characteristic addresses of* $\mathrm{Bif}(W,h)$ *land on the distinguished boundary cycle of E_κ. For parameters in* $\mathrm{Bif}(W,h)$*, the dynamic root has (exact) period n.*
(e) *No two hyperbolic components have a common child component.*

Proof. If $\kappa_0 := \Psi_W(2\pi i h) \in \mathbb{C}$, then κ_0 is a parabolic parameter of period n and rotation number $p/q := h - \lfloor h \rfloor$; in particular, the ray period of κ_0 is qn. Thus W is a parent component of κ_0. By Corollary 5.5, κ_0 is the root of $\mathrm{Bif}(W,h)$, and no other hyperbolic components contain κ_0 on their boundaries. This proves (b); it also proves (a) provided that $\kappa_0 \in \mathbb{C}$. Part (c) was proved in Corollary 5.5.

To prove (d), let $\kappa \in \Gamma_{W,h}$; also choose some parameter $\kappa_1 \in \mathrm{Bif}(W,h)$. Let $\underline{r}$ be the address of a periodic ray landing at a point $z_0 \in \partial U_1$ of the distinguished boundary orbit of E_κ. Recall from Theorem 3.4 that the dynamic root of E_κ is the only periodic point on ∂U_1 which has an essential orbit portrait. Thus the period of $\underline{r}$ is the same as the period of its landing point, which is qn. For $1 \leq j \leq n$, define

$$A_j := \{\sigma^{mn+j-1}(\underline{r}) : 0 \leq m < q\}.$$

Then the landing points of the dynamic rays at the addresses in A_j are those points of the orbit of z_0 which belong to ∂U_j. Similarly as in the proof of Theorem 5.4, we can connect these landing points to the periodic point a_j using attracting dynamic rays such that the resulting "extended rays" $\widetilde{g}_{\sigma^j(\underline{r})}$ do not intersect except in their endpoints. Again, it follows that all addresses in A_1 have the same itinerary under $\widetilde{\underline{s}} := \mathrm{addr}(W,h)$, and this itinerary agrees with that of $\widetilde{\underline{s}}$ in the first $qn-1$ entries. It is easy to see that in fact $\mathcal{O} := \{A_1, \dots, A_n\}$ is an orbit portrait for E_{κ_1}, and that the characteristic rays of this orbit portrait both belong to A_1. (All that needs to be checked is that rays at addresses in different A_j cannot have the same landing point, which follows readily from the way that these rays are permuted by σ^n.) It now follows from Lemma 3.10 that these characteristic rays are in fact the characteristic rays of $\widetilde{\underline{s}}$, and (d) is proved.

To complete the proof of (a), suppose now that $\kappa_0' := \Psi_{\mathrm{Bif}(W,h)}(0) \in \mathbb{C}$. Then by (d) and Theorem 5.3, this parabolic parameter has period n and ray period qn. Thus Corollary 5.5 implies that $\kappa_0' = \kappa_0$.

Finally, let us prove (e). Suppose that $W' = \mathrm{Hyp}(\underline{s}')$, of period m, is a child component of W. By (d), the period n of W is uniquely determined by W'. By the definition of child components,

$$\mathrm{addr}(W) = \sigma^{m-n}(\underline{s}');$$

so $\mathrm{addr}(W)$, and thus W, is uniquely determined by W'. ■

Analysis of Itineraries. We can now describe the addresses of child components of a given hyperbolic component $W = \mathrm{Hyp}(\underline{s})$ in terms of their itineraries under $\underline{s}$. This will be the key to understanding their behavior.

6.3. Lemma (Itineraries of Bifurcation Addresses)**.**
Let $\underline{s}$ be an intermediate external address of length n with kneading sequence $\mathtt{u}_1 \dots \mathtt{u}_{n-1}*$. Furthermore, let $s_* \in \mathbb{Z}$ (resp. $s_* \in \mathbb{Z} + \frac{1}{2}$ if $\underline{s} = \infty$) and $\alpha = p/q \in \mathbb{Q} \cap (0,1)$. If $\underline{s} \neq \infty$, then $\underline{s}' := \mathrm{addr}(\underline{s}, s_*, p/q)$ is the unique intermediate address which satisfies

$$\mathrm{itin}_{\underline{s}}(\underline{s}') = \mathtt{u}_1 \dots \mathtt{u}_{n-1}\mathtt{m}_1\mathtt{u}_1 \dots \mathtt{u}_{n-1}\mathtt{m}_2 \dots \mathtt{u}_1 \dots \mathtt{u}_{n-1}\mathtt{m}_{q-1}\mathtt{u}_1 \dots \mathtt{u}_{n-1}\mathtt{m}_q,$$

where $\mathtt{m}_q = *$, $\mathtt{m}_{q-1} = {}^{s_*}_{s_*-1}$ and

$$\mathtt{m}_j = \begin{cases} s_* & \text{if } j\alpha \in [1-\alpha, 0] \pmod 1 \\ s_* - 1 & \text{otherwise} \end{cases}$$

for $j = 1, \dots, q-2$.

If $\underline{s} = \infty$, then the above is still true, except that $\underline{s}_*$ is replaced by $\underline{s}_* + 1/2$ in the definition of the m_j.

Proof. As in Definition 6.1, let κ be a parameter on the internal ray at angle α in the sector $\mathrm{Sec}(\mathrm{Hyp}(\underline{s}), s_*)$ and let γ be the attracting dynamic ray of E_κ which contains the singular value. Then $\underline{s}' = \mathrm{addr}(\gamma)$. Set $\underline{\tilde{\mathtt{u}}} := \mathrm{itin}_{\underline{s}}(\underline{s}')$.

Since $E_\kappa^{j-1}(\kappa)$ belongs to the Fatou component U_j, it is clear that $\tilde{\mathtt{u}}_{kn+j} = \mathtt{u}_j$ for $k \in \{0 \dots q-1\}$ and $j \in \{1 \dots n-1\}$. So we only need to show that the values $\mathtt{m}_j := \tilde{\mathtt{u}}_{jn}$ have the stated form.

For $j = 1, \dots, q$, let us set $\gamma_j := E_\kappa^{jn-1}(\gamma)$. Note that γ_q is the attracting dynamic ray to $-\infty$ in U_0. Thus $\mathtt{m}_q = *$ and $E_\kappa(\gamma_{q-1})$ is the principal attracting ray. Thus $\mathrm{addr}(\gamma_{q-1}) = s_*\underline{s}$ by definition of s_*; so $\mathtt{m}_{q-1} = {}^{s_*}_{s_*-1}$. Attracting dynamic rays do not intersect each other or their $2\pi i$-translates; thus any other entry $\mathtt{m}_j$ is either s_* or $s_* - 1$, depending on whether γ_j is above or below γ_{q-1}. Since E_κ^n permutes the curves γ_j cyclically with rotation number α, γ_j is above γ_{q-1} if and only if $j\alpha \in [1-\alpha, 1] \pmod 1$. ■

In this and the following section, we will frequently be concerned with the question when two addresses whose itineraries coincide must in fact be the same. Let us therefore state the following simple fact for further reference.

6.4. Observation (Agreeing Itineraries)**.**
Let $\underline{s} \in \overline{\mathcal{S}}$ and $k > 0$. Let $\underline{r}^1, \underline{r}^2$ be addresses with $\sigma^k(\underline{r}^1) \leq \sigma^k(\underline{r}^2)$ whose itineraries (under $\underline{s}$) agree in the first k entries. Suppose furthermore that, for $j = 0, \dots, k-1$, $\sigma^j(\underline{s}) \notin [\sigma^k(\underline{r}^1), \sigma^k(\underline{r}^2)]$.

Then $\underline{r}^1 \leq \underline{r}^2$ and σ^k maps the interval $[\underline{r}^1, \underline{r}^2]$ bijectively onto $[\sigma^k(\underline{r}^1), \sigma^k(\underline{r}^2)]$; in other words, the addresses $\underline{r}^1$ and $\underline{r}^2$ agree in the first k entries.

Proof. Note that it is sufficient to deal with the case $k = 1$; the general case follows by induction.

So suppose that $\underline{r}^1$ and $\underline{r}^2$ are as in the statement, with $k = 1$, and let I denote the interval in $\mathcal{S}$ bounded by $\underline{r}^1$ and $\underline{r}^2$. Since both addresses have the same first itinerary entry, σ maps I bijectively either to $[\sigma(\underline{r}^1), \sigma(\underline{r}^2)]$ or to $\overline{\mathcal{S}} \setminus \big(\sigma(\underline{r}^1), \sigma(\underline{r}^2)\big)$. Since $\underline{s} \notin \sigma(I)$, it follows from the hypotheses that the former must be the case, as required. ■

We can now use this observation to relate the behavior of the itineraries of the addresses $\operatorname{addr}(W, h)$ to that of the addresses themselves.

6.5. Proposition (Monotonicity of Itineraries)**.**
Let $W = \operatorname{Hyp}(\underline{s})$ be a hyperbolic component of period $n \geq 2$, with kneading sequence $\mathbb{K}(W) = \mathtt{u}_1 \dots \mathtt{u}_{n-1}*$.

(a) *Let $I \subset \mathcal{S}$ denote either the interval $[\underline{s}^-, \underline{s}]$ or the interval $[\underline{s}, \underline{s}^+]$ (where $\underline{s}^- < \underline{s}^+$ are, as usual, the characteristic addresses of $\underline{s}$). Suppose that $\underline{r}^1, \underline{r}^2 \in I \setminus \{\underline{s}\}$ have the following property: if $\ell \in \{1, 2\}$ and $j \geq 0$ are such that $\sigma^{jn}(\underline{r}^\ell)$ is defined, then $\sigma^{jn}(\underline{r}^\ell) \in I$ and $\operatorname{itin}_{\underline{s}}(\sigma^{jn}(\underline{r}^\ell))$ starts with $\mathtt{u}_1 \dots \mathtt{u}_{n-1}$. Then*
$$\underline{r}^1 \leq \underline{r}^2 \iff \operatorname{itin}_{\underline{s}}(\underline{r}^1) \leq \operatorname{itin}_{\underline{s}}(\underline{r}^2).$$

(b) *Let $h = \frac{p}{q} \in \mathbb{Q} \setminus \mathbb{Z}$ and $\underline{r} := \operatorname{addr}(W, h)$. If $h > 0$ (resp. if $h < 0$), then $\sigma^{jn}(\underline{r}) \in (\underline{s}^-, \underline{s}]$ (resp. $\sigma^{jn}(\underline{r}) \in [\underline{s}, \underline{s}^+)$) for all $0 \leq j < q$.*

Remark. We should remark on the lexicographic order of itineraries referred to in (a). There is a natural order between integer itinerary entries and boundary symbols: $\mathtt{m} < {}^{\mathtt{j}}_{\mathtt{j}-1}$ if and only if $\mathtt{m} \leq \mathtt{j} - 1$. However, it is not clear how the symbol $*$ should fit into this order. We will fix the convention that the symbol $*$ is incomparable to any other itinerary entry, which gives our claim the strongest possible meaning.

(In fact, this is not relevant for our considerations: the itineraries of any two addresses $\underline{r}^1, \underline{r}^2 \in \mathcal{S}$ will be comparable unless at least one of them is an intermediate external address which is not a preimage of $\underline{s}$. Clearly this cannot happen in our case.)

Proof. To prove item (a); let us fix our ideas by supposing that $I = [\underline{s}^-, \underline{s}]$. As already remarked above, $\operatorname{itin}_{\underline{s}}(\underline{r}^1)$ and $\operatorname{itin}_{\underline{s}}(\underline{r}^2)$ are comparable. Note also that the orbit of $\underline{s}$ does not enter the interval I by the definition of characteristic addresses (this will enable us to apply the previous observation.)

Suppose first that $\operatorname{itin}_{\underline{s}}(\underline{r}^1) = \operatorname{itin}_{\underline{s}}(\underline{r}^2)$. If $\underline{r}^1$ and $\underline{r}^2$ are intermediate, then $\underline{r}^1 = \underline{r}^2$ by Lemma 3.8. On the other hand, if $\underline{r}^1$ and $\underline{r}^2$ are infinite, then for any $\ell \geq 0$, the hypotheses of Observation 6.4 are satisfied with $k = n\ell$. Thus the first $n\ell$ entries of $\underline{r}^1$ and $\underline{r}^2$ agree; since ℓ is arbitrary, this means that $\underline{r}^1 = \underline{r}^2$.

So now suppose that $\operatorname{itin}_{\underline{s}}(\underline{r}^1) \neq \operatorname{itin}_{\underline{s}}(\underline{r}^2)$, say $\operatorname{itin}_{\underline{s}}(\underline{r}^1) < \operatorname{itin}_{\underline{s}}(\underline{r}^2)$. Let $\ell \geq 1$ be such that these itineraries first differ in the $n\ell$-th entry. Then $\underline{r}^1$ and $\underline{r}^2$ agree in the first $(\ell - 1)n$ entries by Observation 6.4. Thus we may suppose, by passing to the $(\ell - 1)n$-th iterates, that $\ell = 1$.

Then $\sigma^{n-1}(\underline{r}^1) < \sigma^{n-1}(\underline{r}^2)$. Since σ^{n-1} preserves the circular order of $\underline{s}$, $\underline{r}^1$ and $\underline{r}^2$, and since $\underline{r}^1, \underline{r}^2 \leq \underline{s}$, it follows that $\underline{r}^1 < \underline{r}^2$, as required.

Now let us prove (b). We again fix our ideas by supposing that $h = p/q > 0$. Let $\mathtt{u}_n$ denote the common n-th itinerary entry of $\underline{s}^-$ and $\underline{s}^+$ with respect to $\underline{s}$. Then the n-th entries of the addresses $\underline{s}^-$ and $\underline{s}^+$ are $\mathtt{u}_n + 1$ and $\mathtt{u}_n$, respectively.

Since $h > 0$, we thus have $s_* \geq \mathtt{u}_n + 1$ by Theorem 4.4. So for $j = 1, \dots, q-1$, the nj-th itinerary entries $\mathtt{m}_j$ of $\underline{r}$ satisfy $\mathtt{m}_j \geq \mathtt{u}_n$ by Lemma 6.3. We claim that, for every $j = 0, \dots, q-2$,

$$\sigma^{(j+1)n}(\underline{r}) \in (\underline{s}^-, \underline{s}] \implies \sigma^{jn}(\underline{r}) \in (\underline{s}^-, \underline{s}].$$

Since $\sigma^{(q-1)n}(\underline{r}) = \underline{s}$, part (b) then follows by induction.

To prove the claim, suppose that $\sigma^{(j+1)n}(\underline{r}) \in (\underline{s}^-, \underline{s}]$. The first $n-1$ itinerary entries of $\underline{s}^-, \sigma^{jn}(\underline{r})$ and $\underline{s}$ are the same, so σ^{n-1} preserves the circular order of these addresses by Observation 2.2, Thus it is sufficient to show that $\sigma^{(j+1)n-1}(\underline{r}) > \sigma^{n-1}(\underline{s}^-)$. This is trivial if $\mathtt{m}_j > \mathtt{u}_n$, and follows from the fact that $\sigma^{(j+1)n}(\underline{r}) \in (\underline{s}^-, \underline{s}]$ if $\mathtt{m}_j = \mathtt{u}_n$. ∎

6.6. Corollary (Monotonicity of $\operatorname{addr}(W, h)$)**.**
Let W be a hyperbolic component. Then the function $h \mapsto \operatorname{addr}(W, h)$ is strictly increasing on each of the intervals $\{h > 0\}$ and $\{h < 0\}$.

Proof. Consider the function $p/q \mapsto \mathtt{m}_1 \dots \mathtt{m}_{q-1}$, where $\mathtt{m}_j$ are the numbers from Lemma 6.3. It is an easy exercise to check that this function is strictly increasing (with respect to lexicographic order).

If $W \neq \operatorname{Hyp}(\infty)$, the claim follows from Proposition 6.5 (a). If $W = \operatorname{Hyp}(\infty)$, the claim follows directly since itineraries and external addresses coincide in this case (compare Remark 2 after Definition 3.5. ∎

6.7. Proposition (Continuity Properties)**.**
Let W be a hyperbolic component and $\underline{s} := \operatorname{addr}(W)$. Then $\lim_{h \to \pm\infty} \operatorname{addr}(W, h) = \underline{s}$. Furthermore, if $h_0 \in \mathbb{R}$, then the behavior of $\operatorname{addr}(W, h)$ for $h \to h_0$ is as follows.

(a) *If $h_0 \in \mathbb{Q} \setminus \mathbb{Z}$, then*

$$\lim_{h \nearrow h_0} \operatorname{addr}(W, h) = \underline{r}^- \text{ and}$$
$$\lim_{h \searrow h_0} \operatorname{addr}(W, h) = \underline{r}^+,$$

where $\underline{r} := \operatorname{addr}(W, h_0)$.

(b) *If $h_0 = 0$ and $\underline{s} \neq \infty$, then*

$$\lim_{h \nearrow 0} \operatorname{addr}(W, h) = \underline{s}^+ \text{ and}$$
$$\lim_{h \searrow 0} \operatorname{addr}(W, h) = \underline{s}^-.$$

(c) *Otherwise, the limit $\operatorname{addr}(W, h_0) := \lim_{h \to h_0} \operatorname{addr}(W, h)$ exists.*

Proof. Let $\mathtt{u}_1 \dots \mathtt{u}_{n-1}*$ be the kneading sequence of W. Let $h_0 = p/q \in \mathbb{Q} \setminus \mathbb{Z}$, and $\underline{r} = \operatorname{addr}(W, h_0)$. Recall that, for parameters $\kappa \in \Gamma_{W,h_0}$, the landing point of the dynamic

ray $g_{\underline{r}^+}$ lies on the distinguished boundary orbit of E_κ. Since the rays landing on this cycle are permuted cyclically with rotation number p/q, it follows easily (as in Lemma 6.3) that

$$\mathrm{itin}_{\underline{s}}(\underline{r}^+) = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n-1}\mathtt{m}_1 \dots \mathtt{u}_1 \dots \mathtt{u}_{n-1}\mathtt{m}_{q-2}\mathtt{u}_1 \dots \mathtt{u}_{n-1}s_*\mathtt{u}_1 \dots \mathtt{u}_{n-1}(s_* - 1)},$$

where the $\mathtt{m}_j$ are as in Lemma 6.3. It follows from Lemma 6.3 that the limit address $\lim_{h \nearrow h_0} \mathrm{addr}(W, h)$ has the same itinerary under $\underline{s}$ as $\underline{r}^+$. If $n = 1$, then the two addresses must trivially be equal. Otherwise, this follows easily from Observation 6.4.

All other parts of the proposition are proved analogously. For each part, we need to prove the equality of two external addresses, at least one of which is given as a monotone limit of addresses $\mathrm{addr}(W, h)$. In each case, it is easy to verify that the corresponding itineraries are equal, which, as above, implies that the same is true for the external addresses. ■

We are now ready to state and prove our main theorem on the structure of the child components of a given hyperbolic component. If $\underline{s} := \mathrm{addr}(W) \neq \infty$, then the *wake* of W is the set $\mathcal{W}(W) := (\underline{s}^-, \underline{s}^+)$; if $\underline{s} = \infty$, $\mathcal{W}(W)$ is defined to be all of $\overline{\mathcal{S}}$. In the following theorem, we use this only for simpler notation, but wakes will play an important role in the next section.

6.8. Theorem (Bifurcation Structure)**.**
Let W be a hyperbolic component and $\underline{s} := \mathrm{addr}(W)$. If $n \geq 2$, then the map $\mathrm{addr}(W, \cdot) : \mathbb{Q} \setminus \mathbb{Z} \to \mathcal{S}$ has the following properties.

(a) $\mathrm{addr}(W, \cdot)$ *is strictly increasing on $\{h > 0\}$ and (separately) on $\{h < 0\}$;*
(b) $\mathrm{addr}(W, \frac{p}{q})$ *is an intermediate external address of length qn (for $\frac{p}{q}$ in lowest terms);*
(c) *if $h \in \mathbb{Q} \setminus \mathbb{Z}$ such that $\Psi_W(2\pi i h) \in \mathbb{C}$, then the parameter $\Psi_W(2\pi i h)$ lies on the boundary of $\overline{\mathrm{Bif}(W, h) = \mathrm{Hyp}}(\mathrm{addr}(W, h))$;*
(d) $\overline{\mathcal{W}(W)} = \overline{\bigcup_{h \in \mathbb{Q}\setminus\mathbb{Z}} \mathcal{W}(\mathrm{Bif}(W, h))}$.
(e) $\lim_{h \to +\infty} \mathrm{addr}(W, h) = \lim_{h \to -\infty} \mathrm{addr}(W, h) = \underline{s}$;
(f) $\lim_{h \nearrow 0} \mathrm{addr}(W, h) = \underline{s}^+$ *and* $\lim_{h \searrow 0} \mathrm{addr}(W, h) = \underline{s}^-$.

These properties uniquely determine the map $\mathrm{addr}(W, \cdot)$, and no such map exists if the preferred parametrization Ψ_W is replaced by some other conformal parametrization $\Psi : \mathbb{H} \to W$ with $\mu \circ \Psi = \exp$.

If $n = 1$, then the map $\mathrm{addr}(W, \cdot)$ is strictly increasing on all of $\mathbb{Q} \setminus \mathbb{Z}$ and satisfies properties (b) to (d) above, and no other map has these properties.

Proof. Let us assume for simplicity that $n > 1$; the proofs for the case $n = 1$ are completely analogous. Property (a) is just the statement of Corollary 6.6, and (b) holds by definition. Property (c) is Proposition 6.2 (a). Properties (e) and (f) were proved in Proposition 6.7.

To establish (d), note first that the inclusion "$\supset$" is clear. To prove "$\subset$", let $\underline{r} \in \overline{\mathcal{W}(W)}$. If $\underline{r} \in \{\underline{s}^-, \underline{s}^+, \underline{s}\}$, then we are done by (e) and (f). Otherwise, there exists $h_0 \in \mathbb{R} \setminus \{0\}$

such that

$$\underline{t}^- := \lim_{h \nearrow h_0} \operatorname{addr}(W,h) \le \underline{r} \le \lim_{h \searrow h_0} \operatorname{addr}(W,h) =: \underline{t}^+.$$

If $h \notin \mathbb{Q} \setminus \mathbb{Z}$, then Proposition 6.7 (c) shows that $\underline{t}^- = \underline{r} = \underline{t}^+$, and we are done. Otherwise, Proposition 6.7 (a) implies that $\underline{r} \in \overline{\mathcal{W}(\operatorname{addr}(W,h))}$, which completes the proof of (d).

Let us now prove the uniqueness statements. Suppose that $a : \mathbb{Q} \setminus \mathbb{Z} \to \mathcal{S}$ also satisfies properties (a) to (d). Then, by item (c) and Corollary 5.5, we know that $a(h) = \operatorname{addr}(W,h)$ whenever $\Psi_W(ih) \in \mathbb{C}$. By Lemma 4.1, the set of such $h \in \mathbb{R}$ is open and dense, so there is a dense set of rationals on which a and $\operatorname{addr}(W,\cdot)$ agree. Properties (a) and (d) then easily imply that Proposition 6.7 is also true for the map a, which shows that both maps must be equal everywhere.

Finally, if the preferred parametrization Ψ_W is replaced by some other parametrization of W, then clearly no map a which satisfies (c) can also satisfy (a). Indeed, such a map must agree with $\operatorname{addr}(W, m + \cdot)$ on a dense set for some $m \in \mathbb{Z} \setminus \{0\}$. Therefore a is not monotone near m by (f). ■

We note the following consequence of the previous theorem for reference.

6.9. Corollary (Subwakes Fill Wake)**.**
Suppose that $\underline{s} \in \mathcal{W}(W) \setminus \{\operatorname{addr}(W)\}$. Then there exists a unique $h \in \mathbb{R}$ such that one (and only one) of the following hold.

(a) $h \in \mathbb{Q} \setminus \mathbb{Z}$ *and* $\underline{s} \in \mathcal{W}(\operatorname{Bif}(W,h))$,
(b) $h \in \mathbb{Q} \setminus \mathbb{Z}$ *and* $\underline{s}$ *is a characteristic address of* $\operatorname{Bif}(W,h)$, *or*
(c) $h \in (\mathbb{R} \setminus \mathbb{Q}) \cup (\mathbb{Z} \setminus \{0\})$ *and* $\underline{s} = \operatorname{addr}(W,h)$.

In particular, if $\underline{s}$ is unbounded, then Property (a) holds. ■

7. Internal Addresses

In this section, we describe the global bifurcation structure of hyperbolic components. The basic question we are now interested in is as follows: given two hyperbolic components V and W, one of which is contained in the wake of the other, how can we determine which bifurcations occur "between" V and W?

We will begin by dividing up the wake of a hyperbolic component W into *sector wakes* $\mathcal{W}(A)$ (one for every sector A of W) such that every child component of W belongs to the wake of the sector from which it bifurcates.

The wakes of two adjacent sectors are separated by a *sector boundary*, i.e. a periodic address which one should think of as the address of the parameter ray landing at the corresponding co-root of W (except that, for now, we do not know whether this co-root parameter actually exists). Every sector A has a natural associated *sector kneading sequence* $\mathbb{K}(A)$ (Definition 7.3). We give a simple description of how kneading sequences depend on the bifurcation structure of parameter space (Theorem 7.7). Finally, we will introduce *internal addresses*, which organize the information encoded in kneading sequences in a "human-readable" way.

Wakes of sectors and combinatoril arcs. Let W be a hyperbolic component, let $h_0 \in \mathbb{Z}$ and consider the sector

$$A = Sec(W, h_0 + frac12) = \Psi_W\big(\{z \in \mathbb{H} : \operatorname{Im} z \in (2\pi h_0, 2\pi(h_0+1))\}\big)$$

of W. The *sector boundaries* of A are defined to be

$$\underline{r}^- := \lim_{h \searrow h_0} \operatorname{addr}(W,h) \quad \text{and} \quad \underline{r}^+ := \lim_{h \nearrow h_0+1} \operatorname{addr}(W,h).$$

(Note that $r^- = \operatorname{addr}(W,h_0)$ unless $h_0 = 0$, in which case r^- is the lower characteristic address of W, and similarly for r^+.) The *wake* of A is denoted by $\mathcal{W}(A) := (\underline{r}^-, \underline{r}^+)$. If $\underline{r}$ is a sector boundary of a sector of A, we also say that $\underline{r}$ is a sector boundary of W.

Armed with this concept, we can introduce a natural (partial) order on sectors and hyperbolic components.

7.1. Definition (Combinatorial Arcs)**.**
If A, B are hyperbolic components or sectors, we write $A \prec B$ if $\mathcal{W}(A) \supset \mathcal{W}(B)$. The combinatorial arc $[A,B]$ *is the set of all hyperbolic components or sectors C such that $A \prec C \prec B$.*

Similarly, if $\underline{s} \in \overline{\mathcal{W}(A)}$, then the combinatorial arc $[A, \underline{s})$ is the set of all C with $A \prec C$ and $\underline{s} \in \overline{\mathcal{W}(C)}$. Note that $[A,\underline{s})$ is linearly ordered by $\prec$.

Remark. We will often also consider open or half-open combinatorial arcs (A,B), $[A,B)$ or $(A,B]$, in which one or both of the endpoints are excluded.

Sector Boundaries and Kneading Sequences.

7.2. Lemma (Sector Boundaries and Itineraries)**.**
Let W be a hyperbolic component of period n and kneading sequence $\mathbb{K}(W) = \mathtt{u}_1 \dots \mathtt{u}_{n-1}*$. Set $\underline{s} := \operatorname{addr}(W)$ and let $\underline{r} \in \mathcal{S}$. Then the following are equivalent.

(a) *$\underline{r}$ is a sector boundary of W;*

(b) $\operatorname{itin}_{\underline{r}}(\underline{s}) = \mathbb{K}(W)$ *and* $\operatorname{itin}_{\underline{s}}(\underline{r}) = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n-1}\mathtt{m}}$ *for some* $\mathtt{m} \in \mathbb{Z}$.

Furthermore, every sector boundary $\underline{r}$ satisfies

$$\sigma^j(\underline{r}) \notin \overline{\mathcal{W}(W)} \tag{4}$$

for $j = 1, \dots, n-1$; in particular, $\underline{r}$ has (exact) period n.

Proof. It follows easily from Lemma 6.3 that for every $\mathtt{m} \in \mathbb{Z}$ there is a sector boundary $\underline{r}$ with $\operatorname{itin}_{\underline{s}}(\underline{r}) = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n-1}\mathtt{m}}$, and every sector boundary $\underline{r}$ has an itinerary of this form. Furthermore, every sector boundary $\underline{r}$ belongs to $\overline{\mathcal{W}(W)}$, and thus satisfies $\operatorname{itin}_{\underline{r}}(\underline{s}) = \mathbb{K}(W)$ by Observation 3.7 (recall that the iterates of $\underline{s}$ do not enter $\mathcal{W}(W)$). By Theorem 3.4, the characteristic addresses of W have period n. In particular, (a) implies (b).

Now let $\kappa \in W$ and let $\underline{r}$ be a sector boundary of W which is not a characteristic address. A simple hyperbolic expansion argument (compare [S2, Proof of Theorem 6.2] or [R1, Theorem 4.2.4]) shows that the dynamic ray $g_{\underline{r}}$ lands on the boundary of the characteristic Fatou component U_1 of E_κ. By the definition of characteristic addresses,

this landing point w is separated from the rest of its orbit by the characteristic rays of E_κ, and its orbit portrait is not essential. In particular, (4) holds (and $\underline{r}$ and w both have exact period n).

To prove that (b) implies (a), suppose that $\underline{r}'$ is an address which is not a sector boundary and has itinerary $\operatorname{itin}_{\underline{s}}(\underline{r}') = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n-1}\mathtt{m}}$. Note that $\underline{r}'$ is necessarily periodic by Lemma 3.8. There is a sector boundary $\underline{r}$ which has the same itinerary as $\underline{r}'$; by Theorem 3.6, the dynamic rays $g_{\underline{r}}$ and $g_{\underline{r}'}$ have a common landing point. By the above, this implies that $\underline{r}$ is a characteristic address of W, and $\underline{r}' \notin \overline{\mathcal{W}(W)}$. By replacing $\underline{r}$ with the other characteristic address of W, if necessary, we may suppose that $\underline{r}$ and $\underline{r}'$ do not enclose $\underline{s}$. It then follows from Observation 6.4 that they must enclose a forward iterate of $\underline{s}$. Thus $\operatorname{itin}_{\underline{r}'}(\underline{s}) \neq \operatorname{itin}_{\underline{r}}(\underline{s}) = \mathbb{K}(\underline{s})$ by Observation 3.7, as required. ■

7.3. Definition and Lemma (Kneading Entries)**.**
Let W be a hyperbolic component of period n and kneading sequence $\mathbb{K}(W) = \mathtt{u}_1 \dots \mathtt{u}_{n-1}*$.

(a) *Let A be a sector of W with sector boundaries $\underline{r}^- < \underline{r}^+$. Then there exists a number $\mathtt{u}(A) \in \mathbb{Z}$ (the* kneading entry *of A) such that*

$$\mathbb{K}^+(\underline{r}^-) = \mathbb{K}^-(\underline{r}^+) = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n-1}\mathtt{u}(A)} =: \mathbb{K}(A).$$

The sequence $\mathbb{K}(A)$ is called the kneading sequence *of the sector A.*

(b) *If $n \geq 2$, then there exists a number $\mathtt{u}(W) \in \{\mathtt{u}_1, \dots, \mathtt{u}_{n-1}\}$ (the* forbidden kneading entry of W*) such that*

$$\mathbb{K}^-(\underline{s}^-) = \mathbb{K}^+(\underline{s}^+) = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n-1}\mathtt{u}(W)} =: \mathbb{K}^*(W)$$

(where $\underline{s}^-$ and $\underline{s}^+$ are the characteristic addresses of W). The sequence $\mathbb{K}^*(W)$ is called the forbidden kneading sequence of W.

The kneading entries of the sectors directly above and below the central internal ray of W are $\mathtt{u}(W) - 1$ and $\mathtt{u}(W) + 1$, respectively. If A and B are any other two adjacent sectors, with A above B, the kneading entries satisfy $\mathtt{u}(A) = \mathtt{u}(B) + 1$ (compare Figure 5(b)).

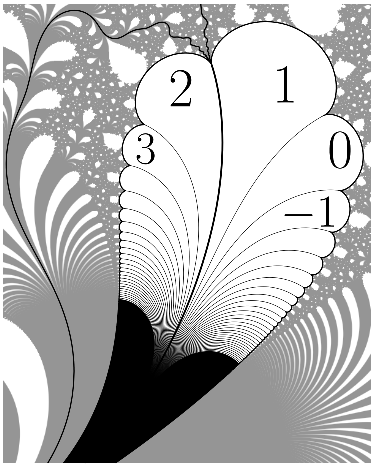
In particular, no two sectors have the same kneading entry and $\mathtt{u}(W)$ is the unique integer which is not assumed as the kneading entry of some sector of W.

(c) *In the period one case, every integer $\mathtt{u}$ is realized as the kneading entry of a sector of W, namely the sector at imaginary parts between $2\pi\mathtt{u}$ and $2\pi(\mathtt{u}+1)$). The period one component thus has no forbidden kneading sequence.*

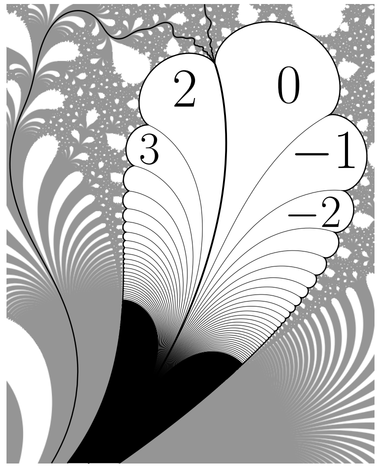
For $\mathtt{m} \neq \mathtt{u}(W)$, we denote the unique sector A satisfying $\mathtt{u}(A) = \mathtt{m}$ by $\operatorname{Sec}(W, \mathtt{m})$.

Proof. In the period one case, the sector boundaries are the addresses $\overline{m}$ with $m \in \mathbb{Z}$, and the claims are trivial. So suppose that $n \geq 2$; to prove (a), let us fix our ideas by supposing that $\underline{r}^- < \underline{r}^+ < \underline{s} := \operatorname{addr}(W)$. It follows from Lemma 6.3 that

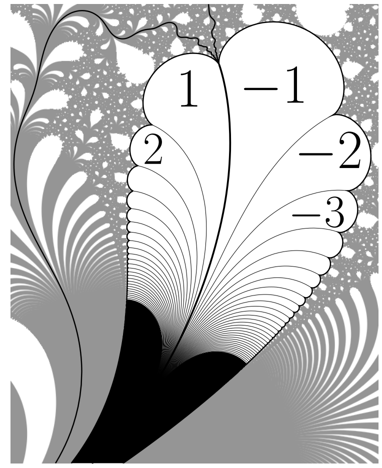
$$\operatorname{itin}_{\underline{s}}(\underline{r}^-) = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n-1}(s_* - 1)} \quad \text{and} \quad \operatorname{itin}_{\underline{s}}(\underline{r}^+) = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n-1}s_*},$$



(a) Sector Labels



(b) Kneading Entries



(c) Sector Numbers

FIGURE 5. Three ways to label sectors, illustrated for the component at address $\underline{s} = 0110\frac{1}{2}\infty$, with $\underline{s}^- = \overline{011002}$ and $\underline{s}^+ = \overline{020101}$. The central internal ray of $\operatorname{Hyp}(\underline{s})$ is emphasized.

where $s_* = s_*(A)$ is the sector label of A. By Observation 3.7 and Lemma 7.2, we see that $\mathbb{K}^+(\underline{r}^-) = \operatorname{itin}_{\underline{s}}(\underline{r}^-)$ and $\mathbb{K}^+(\underline{r}^+) = \operatorname{itin}_{\underline{s}}(\underline{r}^+)$. Since $\underline{r}^+$ is periodic of period n, we thus have

$$\mathbb{K}^+(\underline{r}^-) = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n-1}(s_* - 1)} = \mathbb{K}^-(\underline{r}^+),$$

so $\mathtt{u}(A) = s_* - 1$ is the desired kneading entry.

We already know from Lemma 3.9 that a number $\mathtt{u}(W) \in \mathbb{Z}$ with the required property exists. By Observation 6.4, the $(n-1)$-th iterates of $\underline{s}^-$ and $\underline{s}^+$ must inclose an iterate of $\underline{s}$. Thus the entry $\mathtt{u}(W)$ must occur in $\mathbb{K}(\underline{s})$; i.e. $\mathtt{u}(W) \in \{\mathtt{u}_1, \dots, \mathtt{u}_{n-1}\}$. The remainder of the statement follows from the fact that, for any address $\underline{r}$ of period n, the n-th entries of $\mathbb{K}^-(\underline{r})$ and $\mathbb{K}^+(\underline{r})$ differ exactly by one. ∎

Periodic addresses and hyperbolic components. The last ingredient we require for our analysis of bifurcation structure is the fact that every periodic address is the sector boundary of some hyperbolic component.

7.4. Proposition (Periodic Rays and Intermediate Addresses)**.**
Let $\underline{r} \in \mathcal{S}$ be periodic of period n. Then there exists an intermediate external address $\underline{s}$ of length n such that $\underline{r}$ is a sector boundary of $W := \operatorname{Hyp}(\underline{s})$. If there exists $\widetilde{\underline{r}}$ such that $\langle \underline{r}, \widetilde{\underline{r}} \rangle$ is a characteristic ray pair, then $\underline{r}$ and $\widetilde{\underline{r}}$ are the characteristic addresses of W.

PROOF. Let $\overline{\mathtt{u}_1 \dots \mathtt{u}_{n-1} {}^{\mathtt{u}_n+1}_{\mathtt{u}_n}}$ be the kneading sequence of r and choose a parameter κ for which the singular value lies on the dynamic ray $g_{\underline{r}}$. (It is well-known — and easy to

see — that such parameters exist for every periodic address; see e.g. [BDG].[3] In fact, this is true for every exponentially bounded external address, and these parameters form a corresponding *parameter ray*; see [F, FS, FRS].) Consider the piece of $g_{\underline{r}}$ which connects the singular value with $+\infty$. The dynamic ray $g_{\sigma^{n-1}(\underline{r})} : (0,\infty) \to \mathbb{C}$ is a preimage component of this piece, and thus tends to $+\infty$ for $t \to \infty$, and to $-\infty$ for $t \to 0$. The ray $g_{\sigma^{n-1}}(\underline{r})$ and its translates thus cut the dynamic plane into countably many domains which we call "strips" (similarly as in the definition of itineraries for attracting parameters).

Since $g_{\sigma^{n-1}(\underline{r})} = E_\kappa^{n-1}(g_{\underline{r}})$ tends to $-\infty$ as $t \to 0$, the curve $g_{\underline{r}}$ has an intermediate external address $\underline{s}$ of length n as $t \to 0$ (as well as the usual address $\underline{r}$ as $t \to \infty$). Dynamic rays do not intersect, so both ends of the ray $g_{\underline{r}}$ tend to ∞ within the same strip. In other words, the itineraries of $\underline{r}$ and $\underline{s}$ (with respect to $\underline{r}$) coincide in the first entry. We can apply the same argument to $g_{\sigma(\underline{r})}, \dots, g_{\sigma^{n-2}(\underline{r})}$, and conclude that $\operatorname{itin}_{\underline{r}}(\underline{r})$ and $\operatorname{itin}_{\underline{r}}(\underline{s})$ coincide in the first $n-1$ entries; i.e., $\operatorname{itin}_{\underline{r}}(\underline{s}) = \mathtt{u}_1 \dots \mathtt{u}_{n-1}*$. To fix our ideas, let us suppose that $\underline{r} < \underline{s}$.

Note that for $j = 0, \dots, n-1$, we have $\sigma^j(\underline{s}), \sigma^j(\underline{r}) \notin (\underline{r}, \underline{s})$. Otherwise, we could choose a minimal such j, and Observation 6.4 would imply that σ^j maps $[\underline{r}, \underline{s}]$ to $[\sigma^j(\underline{r}), \sigma^j(\underline{s})] \subset [\underline{r}, \underline{s}]$, which is clearly impossible since no interval is invariant under the shift. By Observation 3.7, it follows that

$$\mathbb{K}(\underline{s}) = \operatorname{itin}_{\underline{r}}(\underline{s}) = \mathtt{u}_1 \dots \mathtt{u}_{n-1} * \quad \text{and} \quad \operatorname{itin}_{\underline{s}}(\underline{r}) = \mathbb{K}^-(\underline{r}) = \overline{\mathtt{u}_1 \dots \mathtt{u}_n}.$$

By Lemma 7.2, $\underline{r}$ is a sector boundary of $W := \operatorname{Hyp}(\underline{s})$.

Finally, suppose that $\widetilde{\underline{r}} \in \mathcal{S}$ is another address of period n such that $\langle \underline{r}, \widetilde{\underline{r}} \rangle$ is a characteristic ray pair. Then $\operatorname{itin}_{\underline{r}}^-(\widetilde{\underline{r}}) = \overline{\mathtt{u}_1 \dots \mathtt{u}_n}$ by Lemma 3.9 (2), and it follows that the cyclic order of $\underline{r}$, $\widetilde{\underline{r}}$ and $\underline{s}$ is preserved by σ^{n-1}. Since $\sigma^{n-1}(\widetilde{\underline{r}}) < \sigma^{n-1}(\widetilde{(\underline{r})})$ and $\sigma^{n-1}(\underline{s}) = \infty$, this means that $\underline{r} < \underline{s} < \widetilde{\underline{r}}$. By Lemma 3.10, $\underline{r}$ and $\widetilde{\underline{r}}$ are the characteristic addresses of $\underline{s}$. ■

Evolution of kneading sequences.

7.5. Lemma (Nested Components Have Different Kneading Sequences)**.**
Let V and W be two hyperbolic components with $W \prec V$. Then $\mathbb{K}(V) \neq \mathbb{K}(W)$.

Proof. This is a direct corollary of Lemma 3.10, "(c) ⇒ a)". ■

7.6. Lemma (Kneading Sequences in Sector Wakes)**.**
Let W be a hyperbolic component, and let A be a sector of W.

(a) *Let $\underline{s} \in \mathcal{W}(A)$ and suppose that $m \geq 1$ is such that $\mathbb{K}(A)$ and $\mathbb{K}(\underline{s})$ have different m-th entries. Then there exists a hyperbolic component $V \in [A, \underline{s})$ of period at most m.*
(b) *Suppose that $V \succ A$ is a hyperbolic component such that there are no components of periods up to m in $[A, V)$. Then $\mathbb{K}^*(V)$ agrees with $\mathbb{K}(A)$ in the first m entries.*
(c) *Suppose that $h \in \mathbb{Q} \setminus \mathbb{Z}$. Then $\mathbb{K}^*(\operatorname{Bif}(W, h)) = \mathbb{K}(\operatorname{Sec}(W, h))$.*

[3]It is possible to formulate the following argument completely combinatorially, without appeal to the existence of such parameters.

(d) *Suppose that* $h \in \mathbb{R} \setminus \mathbb{Q}$. *Then* $\mathbb{K}(\mathrm{addr}(W,h)) = \mathbb{K}(\mathrm{Sec}(W,h))$.

Remark. Note that the addresses in (d) are non-periodic with periodic kneading sequences; compare the remark after Lemma 3.8. (It follows from Theorem 7.7 below that, conversely, these are the only external addresses with this property.)

Proof. Let K consist of all intermediate external addresses of length $\leq m$ in $\mathcal{W}(A)$. Then K is a closed, and hence compact, subset of $\overline{\mathcal{S}}$, and $\mathcal{U} := \{\mathcal{W}(\underline{r}) : \underline{r} \in K\}$ is an open cover of K. Since wakes of hyperbolic components are either nested or disjoint, it follows that $U := \bigcup \mathcal{U}$ has finitely many connected components, each of which is an element of $\mathcal{U}$.

By Proposition 7.4, the set $M := \mathcal{W}(A) \setminus \overline{U}$ contains no periodic addresses of period $\leq m$; thus each of the first m entries of $\mathbb{K}(\underline{s})$ is locally constant when considered as a function of $\underline{s} \in M$. On the other hand, if $(\underline{r}^-, \underline{r}^+) \in \mathcal{U}$, then $\mathbb{K}^-(\underline{r}^-) = \mathbb{K}^+(\underline{r}+)$. Thus the first m kneading sequence entries remain constant throughout M and agree with those of $\mathbb{K}(A)$.

This proves (a). Items (b) through (d) are direct corollaries. ■

We are now ready to prove the main result of this section.

7.7. Theorem (Determining Components on a Combinatorial Arc)**.**
Suppose that A is a sector of a hyperbolic component W, and let $\underline{s} \in \mathcal{W}(A)$. Let j be the index of the first entry at which $\underline{\mathtt{u}} := \mathbb{K}(A)$ and $\underline{\tilde{\mathtt{u}}} := \mathbb{K}(\underline{s})$ differ (or $j = \infty$ if no such entry exists).

(a) *Then there are no hyperbolic components of period less than j on the combinatorial arc $[A, \underline{s})$. If $j < \infty$, then there exists a unique period j component $V \in [A, \underline{s})$. This component has forbidden kneading sequence $\mathbb{K}^*(V) = \overline{\mathtt{u}_1 \dots \mathtt{u}_j}$; if furthermore $\tilde{\mathtt{u}}_j \in \mathbb{Z}$, then $\underline{s} \in \mathcal{W}(\mathrm{Sec}(V, \tilde{\mathtt{u}}_j))$.*

(b) *These statements remain true if $\underline{s}$ and $\underline{\tilde{\mathtt{u}}}$ are replaced by a hyperbolic component W' and its forbidden kneading sequence $\mathbb{K}^*(W')$.*

Proof. To prove (a), let m be the minimal period of a hyperbolic component on $[A, \underline{s})$, and let $V \in [A, \underline{s})$ be a component of period m. (If there are no hyperbolic components in $[A, \underline{s})$, then $j = \infty$ by Lemma 7.6 (a), and there is nothing further to prove.) By Lemma 7.6 (a) and (b), we have $m \leq j$ and $\mathbb{K}^*(V) = \overline{\mathtt{u}_1 \dots \mathtt{u}_m}$. In particular, V is unique by Lemma 7.5.

If $\tilde{\mathtt{u}}_m \notin \mathbb{Z}$, then $\tilde{\mathtt{u}}_m \neq \mathtt{u}_m$, so $m = j$ and we are done. Otherwise, $\underline{s} \in \mathcal{W}(B)$ for some sector B of V. By choice of m and uniqueness of V, there are no components of period $\leq m$ on $[B, \underline{s})$, and it follows from Lemma 7.6 (a) that $\tilde{\mathtt{u}}_m = \mathtt{u}(B) \neq \mathtt{u}(V) = \mathtt{u}_m$; in particular, $m = j$.

Part (b) can be reduced to (a) by choosing $\underline{s}$ to be an address just outside $\mathcal{W}(W)$. ■

Internal addresses. Repeated applications of the preceding theorem enable us to determine the periods and combinatorial order of all hyperbolic components on the combinatorial

arc between $\mathrm{Hyp}(\infty)$ and $\underline{s}$ solely from $\mathbb{K}(\underline{s})$. *Internal addresses*, introduced for Mandel- and Multibrot sets in [LS], organize this information in a more convenient way.

Since neither sector labels nor kneading entries can be easily identified in a picture of parameter space, let us introduce a third labelling method for the sectors of a hyperbolic component. The *sector number* of a given sector is the nonzero integer obtained by counting sectors in counterclockwise orientation, starting at the central internal ray. In other words, the sector number of a sector A is the integer $\mathtt{u}(A) - \mathtt{u}(W) \in \mathbb{Z} \setminus \{0\}$ and can be found in parameter space by counting from the root of W (compare Figure 5(c)).

7.8. Definition (Internal Addresses)**.**
Let $\underline{s} \in \overline{\mathcal{S}}$. Consider the sequence $W_1 = \mathrm{Hyp}(\infty) \prec W_2 \prec \dots$ of all hyperbolic components $W \in [\mathrm{Hyp}(\infty), \underline{s})$ which have the property that all components on $(W, \underline{s})$ have higher period than W. Denote by n_j the period of W_j, and by m_j the sector number of the sector of W_j containing $\underline{s}$. (We adopt the convention that $m_j = \infty$ if $\underline{s} = \mathrm{addr}(W_j)$ and $m_j = k + \frac{1}{2}$ if $\underline{s}$ is the sector boundary with kneading sequence $\mathbb{K}(\underline{s}) = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n_j-1}{}^{\mathtt{u}(W_j)+k+1}_{\mathtt{u}(W_j)+k}}$.)

The internal address of $\underline{s}$ *is defined as the (finite or infinite) sequence*

$$(n_1, m_1) \mapsto (n_2, m_2) \mapsto (n_3, m_3) \mapsto \dots$$

The internal address of an intermediate external address $\underline{s}$ is also called the internal address of the associated hyperbolic component $\mathrm{Hyp}(\underline{s})$*.*

Remark 1. By Theorem 7.7, we could alternatively define W_{j+1} as the unique period of lowest period on the combinatorial arc $(W_j, \underline{s})$. In particular, the components are indeed ordered as stated and the sequence n_j is strictly increasing. (These facts also follow easily directly from the definition.)

Remark 2. Internal addresses do not label hyperbolic components uniquely, reflecting certain symmetries of parameter space: two child components of a given sector with the same denominator but differing numerators of the bifurcation angle have the same internal address. This is the only ambiguity and thus uniqueness can be achieved by specifying bifurcation angles in the internal address, see Theorem A.6.

7.9. Corollary (Computing Internal Addresses)**.**
Two external addresses have the same internal address if and only if they have the same kneading sequence.

Furthermore, the internal address $(1, m_1) \mapsto (n_2, m_2) \mapsto (n_3, m_3) \mapsto \dots$ of any $\underline{s} \in \mathcal{S}$ can be determined inductively from $\underline{\mathtt{u}} := \mathbb{K}(\underline{s})$ by the following procedure:

Set $m_1 := \mathtt{u}_1$. To compute (n_{i+1}, m_{i+1}) from n_i, continue the first n_i entries of $\underline{\mathtt{u}}$ periodically to a periodic sequence $\underline{\mathtt{u}}^i$. Then n_{i+1} is the position of the first difference between $\mathtt{u}$ and $\underline{\mathtt{u}}^i$. Furthermore,

$$m_{i+1} = \begin{cases} \mathtt{u}_{n_{i+1}} - \mathtt{u}^i_{n_{i+1}} & \text{if } \mathtt{u}_{n_{i+1}} \in \mathbb{Z} \\ \infty & \text{if } \mathtt{u}_{n_{i+1}} = * \\ k - \mathtt{u}^i_{n_{i+1}} + \frac{1}{2} & \text{if } \mathtt{u}_{n_{i+1}} = {}^{k+1}_{k}. \end{cases}$$

(If $\mathbb{K}(\underline{s})$ is periodic of period n_{i+1}, or if $\underline{s}$ is intermediate of length n_{i+1}, then the algorithm terminates, and the internal address is finite.)

Remark 1. $\underline{\mathtt{u}}^i$ is the kneading sequence of the sector of W_i containing $\underline{s}$. The forbidden kneading sequence $\mathbb{K}^*(W_i)$ can be obtained by repeating the first n_i entries of $\underline{\mathtt{u}}^{i-1}$ periodically. In particular, if $\underline{s}$ is an intermediate external address, then the forbidden kneading sequence of $\underline{s}$ consists of the first n_{k-1} entries of $\mathbb{K}(\underline{s})$ repeated periodically, where k is the length of the internal address of $\underline{s}$.

Remark 3. As an example, let us consider $\underline{s} = 030\frac{1}{2}\infty$. Then we have $\mathbb{K}(\underline{s}) = \mathtt{0200}*$. Applying the above procedure, we obtain that $\underline{\mathtt{u}}^1 = \overline{0}$, $\underline{\mathtt{u}}^2 = \overline{02}$ and $\underline{\mathtt{u}}^3 = \overline{0200}$, resulting in the internal address

$$(1,0) \mapsto (2,2) \mapsto (4,-2) \mapsto (5,\infty).$$

Remark 3. There is an obvious converse algorithm: given the internal address of $\underline{s}$, we can determine the kneading sequence $\mathbb{K}(\underline{s})$ by inductively defining $\underline{\mathtt{u}}^i$.

Proof. The correctness of the algorithm is an immediate corollary of Theorem 7.7. In particular, the internal address of $\underline{s}$ depends only on $\mathbb{K}(\underline{s})$. Conversely, applying this procedure to two different kneading sequences will produce different internal addresses. ■

We note the following consequences of Theorem 7.7 for further reference.

7.10. Corollary (Combinatorics of Nested Wakes).
Let W and V be hyperbolic components with $W \prec V$. Then all entries of $\mathbb{K}(W)$ also occur in $\mathbb{K}(V)$.

Proof. Let p be the period of W, and let $\mathtt{u}_1 \dots \mathtt{u}_{p-1}*$ be the kneading sequence of W. The proof proceeds by induction on the number n of hyperbolic components on the combinatorial arc (W,V) which have period less than p. If $n = 0$, then it follows from Theorem 7.7 that $\mathbb{K}^*(V)$ begins with $\mathtt{u}_1 \dots \mathtt{u}_{p-1}$, and we are done. (Recall from Lemma 7.3 that all entries of $\mathbb{K}^*(V)$ occur in $\mathbb{K}(V)$.)

If $n > 0$, then let $V' \in (W,V)$ be a component of period $\leq p$. We can now apply the induction hypothesis first to W and V', and then to V' and V. ■

7.11. Corollary (Components on the Combinatorial Arc).
Let $\underline{s} \in \mathcal{S}$ and $\underline{\mathtt{u}} := \mathbb{K}(\underline{s})$. Suppose that $n \geq 1$ such that $\mathtt{u}_n \in \mathbb{Z}$ and $\mathtt{u}_j \in \mathbb{Z} \setminus \{\mathtt{u}_n\}$ for all $j < n$. Then there exists a hyperbolic component W with $\mathbb{K}^*(W) = \overline{\mathtt{u}_1 \dots \mathtt{u}_n}$ and $\underline{s} \in \mathcal{W}(W)$.

Proof. By the internal address algorithm, n appears in the internal address of $\underline{s}$; let V be the associated period n component. The child component of V containing $\underline{s}$ has the required property. ■

Infinitely many essential periodic orbits. To conclude this section, we will give a simple necessary and sufficient criterion for an attracting exponential map to have infinitely many essential periodic orbits. (A non-necessary sufficient condition under which this occurs was the main result of [BD].)

7.12. Proposition (Infinitely Many Essential Orbits)**.**
Let W be a hyperbolic component and $\kappa \in W$. Then the characteristic ray pairs of essential periodic orbits of E_κ are exactly the characteristic ray pairs of hyperbolic components V with $V \prec W$.

In particular, the number of essential periodic orbits of E_κ is finite if and only if the internal address of W is of the form

$$(1, m_1) \mapsto (n_2, m_2) \mapsto (n_3, m_3) \mapsto \dots (n_k, \infty), \tag{5}$$

with $n_j | n_{j+1}$ for all $j < k$. In this case, the number of essential periodic orbits is exactly $k - 1$.

Remark. Using the internal address algorithm, it is simple to convert (5) to a (somewhat more complicated) statement about the kneading sequence of W.

Proof. The first statement follows immediately from Lemma 3.9 and Proposition 7.4. In particular, E_κ has only finitely many essential periodic orbits if and only if W is contained in only finitely many wakes. This is the case if and only if W can be reached by finitely many bifurcations from the period one component $\mathrm{Hyp}(\infty)$, which is exactly what the statement about internal addresses means. ■

Appendix A. Further Topics

In this appendix, we will treat some further developments which are naturally related to the discussion in this article. In the previous sections, we have been careful to give self-contained combinatorial proofs for all presented theorems. With the exception of Theorem A.1, which is again given an independent proof, the subsequent results will not be required in the proofs of the further results referred to in the introduction. Thus, we will often simply sketch how to obtain them from well-known facts in the polynomial setting.

Addresses of Connected Sets. An important application of the results in this article is to obtain control over the combinatorial position of curves and, more generally, connected sets within exponential parameter space.

To make this precise, suppose that $A \subset \mathbb{C}$ is connected and contains at most one attracting or indifferent parameter. Let $\underline{s} \in \mathcal{S}$ and suppose that there exist two hyperbolic components W_1, W_2 with $\mathrm{addr}(W_1) < \underline{s} < \mathrm{addr}(W_2)$ and the following property: there is $R > 0$ such that every component U of

$$\{\mathrm{Re}\, z > R\} \setminus \Big(\Gamma_{W_1,0}\big((-\infty,-1]\big) \cup \Gamma_{W_2,0}\big((-\infty,-1]\big)\Big)$$

which is unbounded but has bounded imaginary parts satisfies $U \cap A = \emptyset$. In this case, we say that A *is separated from* $\underline{s}$. We define

$$\mathrm{Addr}(A) := \{\underline{s} \in \mathcal{S} : A \text{ is not separated from } \underline{s}\}.$$

Note that $\mathrm{Addr}(A)$ is a closed subset of $\mathcal{S}$, and that $\mathrm{Addr}(A)$ is empty if and only if A is bounded.

Remark. If $\gamma : [0, \infty) \to \mathbb{C}$ is a curve to infinity which contains at most one attracting or indifferent parameter, then $\mathrm{Addr}(\gamma)$ consists of a single external address; compare also [RS, Section 2]. In particular, if $G_{\underline{s}}$ is a *parameter ray tail* as defined in [F], then $\mathrm{Addr}(\gamma) = \{\underline{s}\}$.

A.1. Theorem (Addresses of Connected Sets)**.**
Let $A \subset \mathbb{C}$ be connected and contain at most one attracting or indifferent parameter. Then either

(a) *All addresses in* $\mathrm{Addr}(A)$ *have the same kneading sequence, or*
(b) $\mathrm{Addr}(A)$ *consists of the characteristic addresses of some hyperbolic component.*

Proof. We claim that, for any hyperbolic component W, either $\mathrm{Addr}(A) \subset \mathcal{W}(W)$ or $\mathrm{Addr}(A) \cap \mathcal{W}(W) = \emptyset$. In fact, we prove the following stronger fact.

Claim. If $\mathrm{Addr}(A) \cap \mathcal{W}(W) \neq \emptyset$, then there exists $h_0 \in \mathbb{R} \cup \{\infty\}$ such that $\mathrm{Addr}(A) = \{\mathrm{addr}(W, h_0)\}$ if $h \notin \mathbb{Q} \setminus \mathbb{Z}$ and $\mathrm{Addr}(A) \subset \{\mathcal{W}(\mathrm{Bif}(W, h_0))\}$ otherwise. (We adopt the convention that $\mathrm{addr}(W, \infty) = \mathrm{addr}(W)$.)

Proof. Let us suppose that $\mathrm{addr}(W) \neq \infty$ (the proof in the period one case is completely analogous), and suppose that $\underline{s} \in \mathrm{Addr}(A) \setminus \mathrm{addr}(W)$ (if no such address $\underline{s}$ exists, then there is nothing to prove). To fix our ideas, let us assume that $\underline{s} < \mathrm{addr}(W)$.

By Corollary 6.9, there exists $h_0 \in (\mathbb{R} \setminus \{0\})$ such that $\underline{s} = \mathrm{addr}(W, h_0)$ if $h_0 \notin \mathbb{Q} \setminus \mathbb{Z}$ and $\underline{s} \in \mathcal{W}(\mathrm{Bif}(W, h_0))$ otherwise.

By Lemma 4.1, we can choose rational h^- and h^+ with $h^- < h_0 < h_1^+$ arbitrarily close to h_0 such that $\Psi_W(ih^\pm) \in \mathbb{C}$. If A contains a (necessarily unique) indifferent parameter κ_0, then we may suppose that these values are chosen such that $\kappa_0 \notin \Psi_W\big(i[h^-, h^+]\big) \setminus \Psi_W(ih_0)$.

Consider the Jordan arc

$$\gamma := \Gamma_{\mathrm{Bif}(W,h^-),0} \cup \{\Psi_W(ih^-)\} \cup \widetilde{\gamma} \cup \{\Psi_W(ih^+)\} \cup \Gamma_{\mathrm{Bif}(W,h^+),0},$$

where $\widetilde{\gamma} \subset W$ is some curve connecting $\Psi_W(ih^-)$ and $\Psi_W(ih^+)$. Let U denote the component of $\mathbb{C} \setminus \gamma$ which does not contain a left half plane. Since

$$\underline{r}^+ := \mathrm{addr}(W, h^+) > \underline{s} > \underline{r}^- := \mathrm{addr}(W, h),$$

it follows by the definition of $\mathrm{Addr}(A)$ that $U \cap A \neq \emptyset$. Also, $A \cap \gamma = \emptyset$ and A is connected, so $A \subset U$. Therefore A is separated from every address in $\mathcal{S} \setminus [\underline{r}^-, \underline{r}^+]$, and so $\mathrm{Addr}(A) \subset [\underline{r}^-, \underline{r}^+]$. Letting h_1^+ and h_1^- tend to h_0, we have

$$\mathrm{Addr}(A) \subset \big[\lim_{h \nearrow h_0} \mathrm{addr}(W, h)\,,\, \lim_{h \searrow h_0} \mathrm{addr}(W, h)\big],$$

as required.

Let us distinguish two cases.

Case 1: Some address in $\operatorname{Addr}(A)$ *has an infinite internal address.* It then follows from the claim that all addresses have the same internal address, and by Corollary 7.9, they also all share the same kneading sequence.

Case 2: All addresses in $\operatorname{Addr}(A)$ *have a bounded internal address.* It follows from the claim and the definition of internal addresses that there exists some hyperbolic component W such that $\operatorname{Addr}(A) \subset \overline{\mathcal{W}(W)}$ while $\operatorname{Addr}(A)$ is not contained in the wake of any child component of W. Suppose that $\operatorname{Addr}(A)$ contains more than one external address.

If $\operatorname{Addr}(A) \not\subset \mathcal{W}(W)$, then $\operatorname{Addr}(A)$ must consist of the two characteristic addresses of W. If $\operatorname{Addr}(A) \subset \mathcal{W}(W)$, then by the claim there exists $h_0 \in \mathbb{Q} \setminus \mathbb{Z}$ with

$$\operatorname{Addr}(A) \subset \overline{\mathcal{W}(\operatorname{Bif}(W, h_0))} \setminus \mathcal{W}(\operatorname{Bif}(W, h_0)),$$

and again $\operatorname{Addr}(A)$ consists of the two characteristic addresses of $\operatorname{Bif}(W, h_0)$. ■

We record the following special case for use in [FRS] (see there or in [F] for definitions).

A.2. Corollary (Parameter Rays Accumulating at a Common Point)**.**
Suppose that $G_{\underline{s}^1}$ *and* $G_{\underline{s}^2}$ *are parameter rays which have a common accumulation point* κ_0*. Then* $|s^1_j - s^2_j| \leq 1$ *for all* $j \geq 1$*.*

Proof. Let $A := G_{\underline{s}^1} \cup G_{\underline{s}^2} \cup \{\kappa_0\}$. Then $\operatorname{Addr}(A)$ contains $\underline{s}^1$ and $\underline{s}^2$, and by the previous theorem, either $\mathbb{K}(\underline{s}^1) = \mathbb{K}(\underline{s}^2)$ or $\underline{s}^1$ and $\underline{s}^2$ are the characteristic addresses of some hyperbolic component. In either case, the claim follows. ■

In fact, we can sharpen Theorem A.1 to the following statement.

A.3. Theorem (Addresses of Connected Sets II)**.**
Let $A \subset \mathbb{C}$ *be connected and contain at most one attracting or indifferent parameter. Then exactly one of the following holds.*

(a) $\operatorname{Addr}(A)$ *is empty or consists of a single external address.*
(b) $\operatorname{Addr}(A)$ *consists of two bounded external addresses, both of which have the same kneading sequence.*
(c) $\operatorname{Addr}(A)$ *consists of the characteristic addresses of some hyperbolic component.*
(d) $\operatorname{Addr}(A)$ *consists of at least three but finitely many preperiodic addresses. Moreover, there exists a postsingularly finite parameter* $\kappa_0 \in \mathbb{C}$ *such that, for every* $\underline{s} \in \operatorname{Addr}(A)$*, the parameter ray* $G_{\underline{s}}$ *lands at* κ_0*.*

Remark. The *Squeezing Lemma*, proved in [RS], also shows that $\operatorname{Addr}(A)$ cannot contain intermediate or exponentially unbounded addresses.

Sketch of proof. Recall from the proof of Theorem A.1 that all addresses in $\operatorname{Addr}(A)$ are contained in exactly the same wakes, and that the claim is true when their common internal address a is finite. It thus suffices to consider the case where a is infinite.

First suppose that $\operatorname{Addr}(A)$ contains some unbounded infinite external address $\underline{s}$. We need to show that $\operatorname{Addr}(A) = \{\underline{s}\}$. Let $(W_i)_{i\geq 1}$ be the hyperbolic components appearing

in the internal address of $\underline{s}$, and set

$$I := \bigcap_i \overline{\mathcal{W}(W_i)}.$$

Then I is a closed connected subset of $\mathcal{S}$, and $\operatorname{Addr}(A) \subset \{\underline{s}\}$. We claim that I contains no intermediate external addresses (and thus consists of a single point).

Indeed, if $\underline{r} \in I$ was an intermediate external address, then by Corollary 7.11, every entry of $\mathbb{K}(W_i)$ is one of the finitely many symbols of $\mathbb{K}(\underline{r})$. However, $\mathbb{K}(W_i) \to \mathbb{K}(\underline{s})$ by Corollary 7.9, and $\mathbb{K}(\underline{s})$ is unbounded. This is a contradiction.

Now suppose that $\operatorname{Addr}(A)$ consists of more than two external addresses. By the previous step, all addresses in $\operatorname{Addr}(A)$ are bounded; let M be an upper bound on the size of the entries in their (common) kneading sequence, and let $d := 2M + 2$. Then the map

$$\underline{s} \mapsto \sum_{j \geq 1} \frac{s_k}{d^j} \pmod 1$$

takes $\operatorname{Addr}(A)$ injectively to a set $\widetilde{A} \subset \mathbb{R}/\mathbb{Z}$ which has the property that, for any wake of a hyperbolic component W *in the Multibrot set* $\mathcal{M}_d$ *of degree* d, either $\widetilde{A} \subset \mathcal{W}(W)$ or $\widetilde{A} \cap \mathcal{W}(W) = \emptyset$. It follows from the Branch Theorem for Multibrot sets [LS, Theorem 9.1] that $\widetilde{A}$ (and $\operatorname{Addr}(A)$) consists of finitely many preperiodic addresses.

It is easy to see that, in this case, for any $\underline{s}, \underline{r} \in \operatorname{Addr}(A)$, $\operatorname{itin}_{\underline{s}}(\underline{r}) = \mathbb{K}(\underline{s})$. By the main result of [HSS], there exists a parameter κ_0 for which the dynamic ray $g_{\underline{s}}$ lands at the singular value (and thus the singular orbit is finite for this parameter). It follows from [SZ2, Proposition 4.4] that all rays $g_{\underline{r}}$ with $\underline{r} \in \operatorname{Addr}(A)$ also land at the singular value. It follows easily from Hurwitz's theorem and the stability of orbit portraits (compare Proposition 5.2) that all parameter rays $G_{\underline{r}}$ with $\underline{r} \in \operatorname{Addr}(A)$ land at κ_0 (compare [R1, Theorem 5.14.5] or [S1, Theorem IV.6.1] for details). ■

We believe that the ideas of [RS] can be extended to show that that all hypothetical "queer", i.e. nonhyperbolic, components (which conjecturally do not exist) must be bounded; compare the discussion in [RS, Section 8]. The following corollary, which states that such a component could be unbounded in at most two directions, is a first step in this direction.

A.4. Corollary (Nonhyperbolic Components)**.**
Suppose that U is a nonhyperbolic component in exponential parameter space (or more generally, any connected subset of parameter space which contains no attracting, indifferent or escaping parameters). Then $\operatorname{Addr}(U)$ *consists of at most two external addresses.*

Proof. This follows directly from the previous theorem except in the case of item (d). In the latter case, it follows since U cannot intersect any of the parameter rays landing at the associated Misiurewicz points, and thus is separated from all but at most two of these addresses. ■

Angled Internal Addresses. Internal addresses do not label hyperbolic components uniquely. For completeness, we will now discuss a way of decorating internal addresses to restore uniqueness.

A.5. Definition (Angled Internal Address)**.**
Let $\underline{s} \in \mathcal{S}$, and let W_j be the components in the internal address of W_j. Then the angled internal address *of $\underline{s}$ is*

$$(1, h_1) \mapsto (n_2, h_2) \mapsto (n_3, h_3) \mapsto \ldots,$$

where $W_{j+1} \subset \mathcal{W}(\operatorname{Bif}(W_j, h_j))$.

A.6. Theorem (Uniqueness of Angled Internal Addresses)**.**
No two hyperbolic components share the same angled internal address.

Sketch of proof. Suppose that $W_1 \neq W_2$ have the same angled internal address a. Let M be an upper bound for the entries of $\mathbb{K}(W_1) = \mathbb{K}(W_2)$, and set $d := 2M + 2$. Similarly as in the proof of Theorem A.3, it then follows that the Multibrot Set $\mathcal{M}_d$ contains two different hyperbolic components $\widetilde{W_1}$ and $\widetilde{W_2}$ which both have the same angled internal address. This contradicts [LS, Theorem 9.2]. □

A Combinatorial Tuning Formula. Let $\underline{s}$ be an intermediate external address of length ≥ 2. We will give an analog of the concept of tuning for polynomials, on a combinatorial level. For every i, let us denote by r_{i-1}^{i} the first n entries of the sector boundary $\operatorname{Bdy}(\underline{s}, {}_{i-1}^{\,i})$.

A map $\tau : \overline{\mathcal{S}} \to \mathcal{W}(\underline{s})$ is called a tuning map for $\underline{s}$, if $\tau(-\infty) = \underline{s}$ and

$$\tau(k\underline{r}) = \begin{cases} r_{\mathtt{u}_n+k-1}^{\mathtt{u}_n+k}\tau(\underline{r}) & \tau(\underline{r}) > \underline{s} \\ r_{\mathtt{u}_n+k}^{\mathtt{u}_n+k+1}\tau(\underline{r}) & \tau(\underline{r}) < \underline{s} \\ r_{\mathtt{u}_n+k-\frac{1}{2}}^{\mathtt{u}_n+k+\frac{1}{2}}\tau(\underline{r}) & \tau(\underline{r}) = \underline{s}. \end{cases}.$$

There are exactly two such maps, which are uniquely defined by choosing $\tau(\overline{0})$ to be either $\overline{r_{\mathtt{u}_n}^{u_n+1}}$ or $\overline{r_{\mathtt{u}_n-1}^{\mathtt{u}_n}}$. (Note that under a tuning map, some addresses which are not exponentially bounded will be mapped to addresses which are exponentially bounded. This is related to the fact that topological renormalization fails for exponential maps; see [R1, Section 4.3] or [R3].)

A.7. Theorem (Tuning Theorem)**.**
If the internal address of $\underline{r}$ is $(1, m_1) \mapsto (n_2, m_2) \mapsto (n_3, m_3) \mapsto \ldots$, and the internal address of $\underline{s}$ is $(1, \widetilde{m}_1) \mapsto (\widetilde{n}_2, \widetilde{m}_2) \mapsto \ldots \mapsto (n, \infty)$, then the internal address of $\tau(\underline{r})$ is

$$(1, \widetilde{m}_1) \mapsto (\widetilde{n}_2), \widetilde{m}_2) \mapsto \ldots \mapsto (n, m_1') \mapsto (n * \widetilde{n}_2, m_2) \mapsto (n * \widetilde{n}_3, m_3) \mapsto \ldots,$$

where m_1' is $m_1 + 1$ or m_1, depending on whether $m_1 \geq 0$ or $m_1 < 0$.

Sketch of proof. This can be easily inferred from the well-known tuning formula for Multibrot sets (see e.g. [M3, Theorem 8.2] or [LS, Proposition 6.7]).

Alternatively, it is not difficult to give a direct combinatorial proof of this fact; see [R1, 5.11.2]. □

Appendix B. Combinatorial Algorithms

In this article, we have seen several ways to describe hyperbolic components, and our results allow us to compute any of these from any other. Since these algorithms have not always been made explicit in the previous treatment, we collect them here.

First note the kneading sequence of a hyperbolic component W can be easily computed from its intermediate external address according to the definition (Definition 3.5). How to compute the internal address of a hyperbolic component from its kneading sequence was shown in Corollary 7.9, which also describes how to obtain the forbidden kneading sequence $\mathbb{K}^*(W)$ from $\mathbb{K}(W)$. Furthermore, $\mathbb{K}^*(W)$ can easily be computed from its characteristic ray pair $\langle \underline{s}^-, \underline{s}^+\rangle$.

Algorithm B.1 (Computing $\operatorname{addr}(W)$ given $\langle \underline{s}^-, \underline{s}^+\rangle$)**.**
Given: *Kneading sequence* $\mathbb{K}(W) = \mathtt{u}_1 \dots \mathtt{u}_{n-1}*$ *and some* $\underline{r} \in \overline{\mathcal{W}(W)}$.

Aim: *Compute* $\operatorname{addr}(W)$.

Algorithm: *Let* $\underline{s}^n := \infty$*, and compute* $\underline{s}^{n-1}, \dots, \underline{s}^1$ *inductively by choosing* $\underline{s}^{j-1}$ *to be the unique preimage of* $\underline{s}^j$ *in* $\big(\mathtt{u}_{j-1}\underline{r}, (\mathtt{u}_{j-1}+1)\underline{r}\big)$*. Then* $\operatorname{addr}(W) = \underline{s}^1$.

Proof. By Observation 3.7 and Lemma 3.8, $\operatorname{addr}(W)$ is the unique address $\underline{s}$ with $\operatorname{itin}_{\underline{r}}(\underline{s}) = \mathbb{K}(W)$. This is exactly the address $\underline{s}^1$ computed by this algorithm. ■

Algorithm B.2 (Computing $\langle \underline{s}^-, \underline{s}^+\rangle$ given $\operatorname{addr}(W)$)**.**
Given: *Kneading sequence* $\mathbb{K}(W) = \mathtt{u}_1 \dots \mathtt{u}_{n-1}*$*, some* $\underline{r} \in \overline{\mathcal{W}(W)}$ *and some* $s_* \in \mathbb{Z}$.

Aim: *Compute* $\operatorname{Bdy}\big(W, {}^{s_*}_{s_*-1}\big)$.

Algorithm: *Compute the unique preimage* $\widetilde{\underline{r}}$ *of* $s_*\underline{r}$ *whose itinerary (with respect to* $\underline{r}$*) begins with* $\mathtt{u}_1 \dots \mathtt{u}_{n-1}$*, as in Algorithm B.1. The sought address is obtained by continuing the first* n *entries of* $\widetilde{\underline{r}}$ *periodically.*

Remark. To actually compute the characteristic addresses of W, first compute the forbidden kneading entry $\mathtt{u}_n = \mathtt{u}(W)$, and then apply the algorithm to $s_* = \mathtt{u}_n$ and $s_* = \mathtt{u}_n + 1$.

Proof. Using Observation 6.4 (similarly to Proposition 6.5 (a), one shows that the interval between $\widetilde{\underline{r}}$ and the required address is mapped bijectively by σ^n. ■

Algorithm B.3 (Compute the Angled Internal Address of W)**.**
Given: *An intermediate external address* $\underline{s}$.

Aim: *Compute the angled internal address of* $\underline{s}$.

Algorithm: *Let* n *be the length of* $\underline{s}$*, and calculate the kneading sequence* $\underline{\mathtt{u}} = \mathtt{u}_1 \dots \mathtt{u}_{n-1}* :=$ $\mathbb{K}(\underline{s})$*. Set* $n_1 := 1$ *and* $m_1 := \mathtt{u}_1$.

Given, for some $j \geq 1$, two numbers $n_j < n$ and $p^j \in \mathbb{Z}$, we calculate three numbers $n_{j+1} > n_j$, $m_{j+1} \in \mathbb{Z}$ and $h_j \in \mathbb{Q}$. The algorithm terminates when n_{j+1} is equal to n; at this point the angled internal address of $\underline{s}$ will be given by

$$(n_1, h_1) \mapsto (n_2, h_2) \mapsto \ldots \mapsto (n_j, h_j) \mapsto (n, \infty).$$

Step 1: Calculation of n_{j+1} and the corresponding sector number. *Define n_{j+1} to be the first index at which $\underline{\mathtt{u}}$ and $\underline{\mathtt{u}}^j := \overline{\mathtt{u}_1 \ldots \mathtt{u}_{n_j}}$ differ. If $n_{j+1} \neq n$, then set $m_{j+1} := \mathtt{u}_{n_{j+1}} - \mathtt{u}^j_{n_{j+1}}$.*

Step 2: Determining the denominator. *Let us inductively define a finite sequence $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_r$ as follows. Set $\ell_1 := n_{j+1}$; if n_j does not divide ℓ_k, let ℓ_{k+1} be the first index at which $\underline{\mathtt{u}}^j$ and $\overline{\mathtt{u}_1 \ldots \mathtt{u}_{\ell_k}}$ differ. Otherwise, we terminate, setting $r := k$. Set $q_j := \ell_r / n_j$.*

Step 3: Determining the numerator. *If $q_j = 2$, set $p_j := 1$. Otherwise, calculate*

$$x := \#\{k \in \{2, \ldots, q_j - 2\} : \sigma^{kn_j}(\underline{s}) \text{ lies between } \underline{s} \text{ and } \sigma(\underline{s})\}.$$

Set $p_n := x + 1$ if $\sigma(\underline{s}) > \underline{s}$, and $p_n := n_j - x - 1$ otherwise.

Step 4: Determining h_j. *h_j is defined to be $m_j + p_n/q_n$ if $j = 1$ or $m_j < 0$, and $m_j - 1 + p_n/q_n$ otherwise.*

Proof. Let W_j denote the j-th component in the internal address of $\underline{s}$. By Corollary 7.9, the value of n_j computed by our algorithm will be the period of W_j, and if $n_j \neq n$, then m_j is the number of the sector containing $\underline{s}$.

If $n_j \neq n$, let V_j be the child component of W containing $\underline{s}$. By Theorem 7.7, this component has period $\ell_r = q_j n_j$. By Proposition 6.5 (a), the interval $\mathcal{W}(V_j)$ is mapped bijectively by $\sigma^{(q_j - 2)n_j}$. (Note that this shows, in particular, that $\ell_r < 2n_j + n$, and thus limits the iterations necessary in Step 2.) Therefore, the order of the iterates of $\underline{s}$ and those of $\operatorname{addr}(V_j)$ is the same. It follows that $V_j = \operatorname{Bif}(W_j, h_j)$, where h_j is defined as indicated. ∎

Remark. In Step 3, we could have instead first calculated the intermediate external address of the bifurcating component in question (using Algorithm B.4 below), and then calculated its rotation number. This introduces an extra step in the algorithm, but makes the proof somewhat simpler.

It remains to indicate how an intermediate external address can be recovered from its angled internal address. Let us first note how to handle the special case of computing a combinatorial bifurcation.

Algorithm B.4 (Computing Combinatorial Bifurcations)**.**
Given: $\underline{s} := \operatorname{addr}(W)$, *$s_* \in \mathbb{Z}$ and $\alpha = \frac{p}{q} \in (0,1) \cap \mathbb{Q}$.*

Aim: *Compute* $\operatorname{addr}(W, s_*, \alpha)$.

Algorithm: *Compute the unique address whose itinerary under $\underline{s}$ is as given by Lemma 6.3. (Alternatively, compute the sector boundaries of the given sector and apply the Combinatorial Tuning Formula.)* ∎

Algorithm B.5 (Computing $\operatorname{addr}(W)$ from an angled internal address)**.**
Given: *The angled internal address $a = (n_1, h_1) \mapsto \ldots \mapsto (n, \infty)$ of some hyperbolic component W.*

Aim: *Compute the unique intermediate external address $\underline{s}$ whose angled internal address is a.*

Algorithm: *Let W_i denote the component represented by the i-th entry of a. We will compute $\underline{s}^i := \operatorname{addr}(W_i)$ inductively as follows; note that $\underline{s}^1 = \infty$.*

Compute the upper characteristic address $\underline{r}^+$ of the component $V_i := \operatorname{Bif}(W_i, h_i,)$ (by applying Algorithms B.4 and B.2, or by using the combinatorial tuning formula).

Let $\underline{t}^1 \leq \underline{r}^+$ be maximal such that $\underline{t}^1$ is periodic of period at most n_{i+1}. If the period of $\underline{t}^1$ is strictly less than n_{i+1}, then $\underline{t}^1$ is the upper characteristic address of some hyperbolic component. Compute the lower characteristic address $\underline{t}^{1-}$ of this component and let $\underline{t}^2 \leq \underline{t}^{1-}$ be maximal such that $\underline{t}^2$ is periodic of period at most n_{i+1}.

Continue until an address $\underline{t}^k$ is computed which is periodic of period n_{i+1}. This is the upper characteristic addresses of $\underline{s}^{i+1}$, and we can compute $\underline{s}^i$ using Algorithm B.1.

Proof. Recall that $\underline{s}^{i+1}$ is not contained in the wake of any component $U \prec V_i$ which has smaller period than n_{i+1}, and by the uniqueness of angled internal addresses (Theorem A.6), there are only finitely many periodic addresses $\underline{t}^j$ which are encountered in each step. Thus, the algorithm will indeed terminate and compute an address which is periodic of period n_{i+1}. The associated hyperbolic component then has angled internal address

$$(n_1, h_1) \mapsto \ldots \mapsto (n_j, h_j) \mapsto (n_{j+1}, \infty)$$

and the claim follows by Theorem A.6. ■

Finally, let us mention that it is very simple to decide whether a given hyperbolic component is a child component, and to calculate the address of the parent component in this case. In particular, it is easy to decide whether two given hyperbolic components have a common parabolic boundary point (or, in fact, *any* common boundary point, compare [RS, Proposition 8.1]).

Algorithm B.6 (Primitive and Satellite Components)**.**
Let $\underline{s} \in \mathcal{S}$ be an intermediate external address with kneading sequence $\mathtt{u}_1 \ldots \mathtt{u}_{n-1}*$. If there exists some $\mathtt{u}_n \in \mathbb{Z}$ such that $\overline{\mathtt{u}_1 \ldots \mathtt{u}_{n-1}\mathtt{u}_n}$ is periodic with period $j < n$, then $\operatorname{Hyp}(\underline{s})$ is a child component of $\operatorname{Hyp}(\sigma^{n-j}(\underline{s}))$.

Otherwise, $\operatorname{Hyp}(\underline{s})$ is a primitive component.

Proof. This is an immediate consequence of Corollary 5.5 and Theorem 6.8. ■

List of Symbols

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References

[BR] I. Noël Baker and Philip J. Rippon, *Iteration of exponential functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 49–77.

[BD] Ranjit Bhattacharjee and Robert L. Devaney, *Tying hairs for structurally stable exponentials*, Ergodic Theory Dynam. Systems **20** (2000), no. 6, 1603–1617.

[BDD] Ranjit Bhattacharjee, Robert L. Devaney, R. E. Lee Deville, Krešimir Josić, and Monica Moreno-Rocha, *Accessible points in the Julia sets of stable exponentials*, Discrete Contin. Dyn. Syst. Ser. B **1** (2001), no. 3, 299–318.

[BDG] Clara Bodelón, Robert L. Devaney, Michael Hayes, Gareth Roberts, Lisa R. Goldberg, and John H. Hubbard, *Hairs for the complex exponential family*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **9** (1999), no. 8, 1517–1534.

[DFJ] Robert L. Devaney, Núria Fagella, and Xavier Jarque, *Hyperbolic components of the complex exponential family*, Fund. Math. **174** (2002), no. 3, 193–215.

[DH] Adrien Douady and John Hubbard, *Etude dynamique des polynômes complexes*, Prépublications mathémathiques d'Orsay (1984 / 1985), no. 2/4.

[ES] Dominik Eberlein and Dierk Schleicher, *Rational parameter rays of Multibrot sets*, in preparation, compare Thesis 1999-02 on the Stony Brook Thesis Server.

[EL1] Alexandre È. Eremenko, *On the iteration of entire functions*, Dynamical systems and ergodic theory (Warsaw, 1986), Banach Center Publ., vol. 23, PWN, Warsaw, 1989, pp. 339–345.

[EL2] Alexandre È. Eremenko and Mikhail Yu. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020.

[F] Markus Förster, *Parameter rays for the exponential family*, Diplomarbeit, Techn. Univ. München, 2003, Available as Thesis 2003-03 on the Stony Brook Thesis Server.

[FRS] Markus Förster, Lasse Rempe, and Dierk Schleicher, *Classification of escaping exponential maps*, Preprint, 2003 arXiv:math.DS/0311427; submitted.

[FS] Markus Förster and Dierk Schleicher, *Parameter rays for the exponential family*, Manuscript, 2004.

[HSS] John H. Hubbard, Dierk Schleicher and Mitsuhiro Shishikura, *A topological characterization of postsingularly finite exponential maps and limits of quadratic differentials*, Manuscript, 2004.

[LS] Eike Lau and Dierk Schleicher, *Internal addresses in the Mandelbrot set and irreducibility of polynomials*, Preprint #1994/19, Institute for Mathematical Sciences, SUNY Stony Brook, 1994, arXiv:math.DS/9411238.

[M1] John Milnor, *Dynamics in one complex variable*, Friedr. Vieweg & Sohn, Braunschweig, 1999, arXiv:math.DS/9201272.

[M2] ———, *On rational maps with two critical points*, Experiment. Math. **9** (2000), no. 4, 481–522, arXiv:math.DS/9709226.

[M3] ———, *Periodic orbits, externals rays and the Mandelbrot set: an expository account*, Astérisque (2000), no. 261, xiii, 277–333, arXiv:math.DS/9905169, Géométrie complexe et systèmes dynamiques (Orsay, 1995).

[PR] Carsten L. Petersen and Gustav Ryd, *Convergence of rational rays in parameter spaces*, The Mandelbrot set, theme and variations, London Math. Soc. Lecture Note Ser., vol. 274, Cambridge Univ. Press, Cambridge, 2000, pp. 161–172.

[R1] Lasse Rempe, *Dynamics of exponential maps*, doctoral thesis, Christian-Albrechts-Universität Kiel, 2003, `http://e-diss.uni-kiel.de/diss_781/`.

[R2] ———, *A landing theorem for periodic rays of exponential maps (to appear)*, Proc. Amer. Math. Soc (2005), arXiv:math.DS/0307371.

[Rr] ———, *Topological dynamics of exponential maps on their escaping sets*, Preprint, 2003, arXiv:math.DS/0309107, conditionally accepted for publication in Ergodic Theory Dynam. Systems.

[R3] ———, *Nonlanding dynamic rays of exponential maps*, Manuscript, 2004.

[RS] Lasse Rempe and Dierk Schleicher, *Bifurcations in the space of exponential maps*, Preprint #2004/03, Institute for Mathematical Sciences, SUNY Stony Brook, 2004, arXiv:math.DS/0311480.

[S1] Dierk Schleicher, *On the dynamics of iterated exponential maps*, Habilitation thesis, TU München, 1999.

[S2] ______, *Attracting dynamics of exponential maps*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 3–34.

[S3] ______, *Hyperbolic components in exponential parameter space*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 223–228.

[S4] ______, *On fibers and local connectivity of Mandelbrot and Multibrot sets*, Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot (Michel Lapidus and Machiel van Frankenhuysen, eds.), Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc., to appear, arXiv:math.DS/9902155.

[SZ1] Dierk Schleicher and Johannes Zimmer, *Escaping points of exponential maps*, J. London Math. Soc. (2) **67** (2003), no. 2, 380–400.

[SZ2] ______, *Periodic points and dynamic rays of exponential maps*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 327–354.

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